1 Additive and Multiplicative Systems

We have discussed positional systems, which use a handful of symbols and relative positions of these symbols to determine the numbers they represent. Now, we are going to talk about two other types of numeration systems, and we will see two ancient cultures that implemented these systems.

1.1 Additive systems: the Egyptian numeration system

We have used the term numeration system, but we have not as of yet gave a formal definition.

Definition 1 *A* numeration system *is a set of symbols, called* numerals*, together with a set of rules for writing these numerals to represent numbers.*

For example, our Hindu-Arabic system is a numeration system. The set of numerals we use are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and the rules we use instruct us how to put these numerals into different positions to represent different numbers. Of course, positional systems are only one type of numeration system.

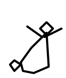
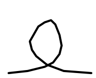
Definition 2 *An* additive numeration system *is a numeration system in which the number represented by the written numerals is simply the sum of the values each numeral represents.*

Perhaps an example is appropriate.

Example 3 *Suppose we use an additive system to represent numbers, and we have as numerals $A = 1$ and $B = 10$. Then $BBAAA$ would represent 23. This is because there are two B's, each of which represent 10, and there are 3 A's, each of which represent 1. Adding, we have* $10 + 10 + 1 + 1 + 1 = 23$.

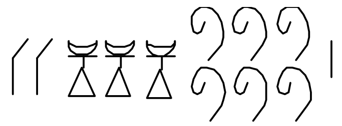
Example 4 *Using the same numeration system as the last example, $ABBABBAAAB$ would represent* $1 + 10 + 10 + 1 + 10 + 10 + 1 + 1 + 1 + 10 = 55$.

The previous example indicates that, in an additive system, the order of the numerals is unimportant. Again, we simply add the values that each numeral represents. Where that numeral is located in relation to the other numerals is irrelevant.
The Egyptians implemented an additive system. They used the following numerals.

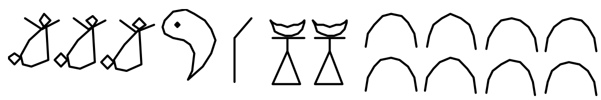
	= 1	stroke
	= 10	heel bone
	= 100	scroll
	= 1000	lotus flower
	= 10,000	pointing finger
	= 100,000	polywog
	= 1,000,000	astonished man
	= 10,000,000	rising sun?

Example 5 *Write 23,602 in Egyptian numerals.*

We use two pointing fingers to represent 20,000, three louts flowers to represent 3,000, 6 scrolls to represent 600, and 2 strokes to represent 2. We get



Again, the order of these symbols is irrelevant; we "stacked" the scrolls in the above numeral only to save space. However, even though the order is irrelevant, we typically write the numerals in descending values from left to right, if only to mimic how we write numerals in the Hindu-Arabic system. In fact, the Egyptians wrote most of their numerals in ascending values from left to right, if they were in fact writing from left to right at the time. Sometimes they would write from right to left, sometimes they would write from top to bottom, and sometimes they would write from bottom to top. It all depended on what space was available.

Example 6 *Write*  *in Hindu-Arabic.*

There are three astonished men, representing 3,000,000; there is one polywog, representing 100,000; there is one pointing finger, representing 10,000; there are two lotus flowers, representing 2,000; there are eight heel bones, representing 80. So, the number represented by this Egyptian numeral is $3,112,080$.

1.2 Multiplicative systems: the Chinese numeration system

In some respects, a multiplicative numeration system has characteristics of both additive systems and positional systems. For example, in our Hindu-Arabic system, 23 represents $2 \times 10 + 3$. Note that we have to add the 2×10 and the 3 to get the value of the number represented, much like we do in additive systems. We do much the same thing in a multiplicative system, except that instead of the position dictating what is multiplied, an explicit symbol is used to represent multiplication is to be performed.

In the above example, the "2" and "3" represent the digits of the number, and "10" represents the position of the "2." In a multiplicative system, we would have symbols for each of these digits, and we would have a symbol for the "10." We would write the "2" and "10" together to represent 2×10, and we would then write the "3" afterward. Then we would add the values of 2×10 and 3 together to get the value of 23.

Definition 7 *A* multiplicative numeration system *consists of two sets of numerals. The elements in one of these sets represent "digits," and the elements in the other set represent "positions." If necessary, we use a digit symbol and a position symbol together, and we multiply the values of the individual numerals to get the number represented. Finally, after any necessary multiplications are done, we add the values together to get the final number being represented.*

Example 8 *Suppose our digits are* $\{1, 2, 3, 4\}$ *and our positions are* $\{A, B, C\}$, *where A represents 10, B represents 100, and C represents 1000. If this is a multiplicative system, how would we represent the Hindu-Arabic numeral* 4231?

Writing 4231 in expanded notation, we see $4132 = 4 \times 1000 + 2 \times 100 + 3 \times 10 + 1$. The symbol for 1000 is C, the symbol for 100 is B, and the symbol for 10 is A. So, using our multiplicative system, we have $4132 = 4C2B3A1$.

Example 9 *Using the same system as the previous example, what Hindu-Arabic numeral is represented by* $3B2A4$?

As before, B represents 100, and A represents 10. Since A and B are "position" symbols, we multiply them by the digits to find the values they represent, and then add all the values together. So, $3B = 3 \times 100 = 300$ and $2A = 2 \times 10 = 20$. So, $3B2A4 = 300 + 20 + 4 = 324$.

The Chinese implemented a multiplicative numeration system. They used the following digit and position numerals.

digits	positions
一 = 1	十 = 10
二 = 2	百 = 100
三 = 3	千 = 1000
四 = 4	
五 = 5	
六 = 6	
七 = 7	
八 = 8	
九 = 9	

Example 10 *Represent* 7325 *using Chinese numerals.*

Using expanded notation, $7325 = 7 \times 1000 + 3 \times 100 + 2 \times 10 + 5$. To represent 7×1000, we write the numeral for 7 directly followed by the numeral for 1000. To represent 3×100, we write the numeral for 3 directly followed by the numeral for 100. To represent 2×10, we write the numeral for 2 directly followed by the numeral for 10. Finally, to represent 5, we write the numeral for 5. We then "string" all these symbols together to represent the final number. The Chinese customarily wrote their numerals vertically; we will follow this custom. So, our final Chinese numeral is

七千三百二十五

Example 11 *Write*

六千九百八

in Hindu-Arabic.

We have the numeral for 6 directly followed by the numeral for 1000. So, this portion represents 6×1000. We also have the numeral for 9 directly followed by the numeral for 100. So, this portion represents 9×100. Finally, we have the numeral for 8. Since it is not followed by any other numeral, it simply represents 8. So, the final value is $6 \times 1000 + 9 \times 100 + 8 = 6908$.

Example 12 *Write 2130 in Chinese.*

This example has a bit of a twist. In expanded notation, we have $2130 = 2 \times 1000 + 1 \times 100 + 3 \times 10$. To represent 2000, we use the numeral for 2 followed the numeral for 1000, and to represent 30, we use the numeral for 3 followed by the numeral for 10. However, to represent 100, we only write the symbol for 100. We know $1 \times 100 = 100$, so writing the numeral for 1 and then the numeral for 100 would be redundant. So, our Chinese numeral for 2130 would be

二
千
百
三
十