1 Converting from one base to another

1.1 Positional systems

Our numeration system, called the *Hindu-Arabic system*, is based on the number 10. Other cultures have used different bases for their positional systems. As we have seen, the Mayans used a base 20 system. The Babylonians used a base 60 system. Other popular bases for numeration systems are 5, 8, and 12. (Think about why!)

In a positional system, the position of each digit in the number is important. Let's first consider our system, a base 10 system. For example, in the number 432, the "2" represents two 1's. In the number 423, the "2" represents two 10's. In the number 234, the "2" represents two 100's. In general, if we look at the number

$$...abcd.efgh...$$

the position immediately to the left of the decimal point is the 1's position. So, the digit d represents that there are d 1's. The next position to the left is the 10's position. So, the digit c represents that there are c 10's. The next position to the left is the 100's position. So, the digit b represents that there are b 100's. The next position to the left is the 1000's position. So, the digit a represents that there are a 1000's. The dots to the left of the a represent that there could be additional digits to the left; continuing from the a, those places would represent 10,000's, 100,000's, 1,000,000's, etc.

Similarly, we can look at the positions starting immediately to the right of the decimal point. That position is the $\frac{1}{10}$'s position. So, the digit e represents that there are e $\frac{1}{10}$'s. The next position to the right is the $\frac{1}{100}$'s position. So, the digit f represents that there are f $\frac{1}{100}$'s. Again, we can continue this process indefinitely to the right. The next position is the $\frac{1}{1000}$'s position, the next is the $\frac{1}{10000}$'s position, and so on.

Let's do a specific example. Consider the number 3196.403. The 3 represents that there are three 1000's, meaning the actual value of the 3 is 3000, or 3×1000. The 1 represents that there is one 100's, meaning the actual value of the 1 is 100, or 1×100. Continuing, we see the actual value of the 9 is 90, or 9×10, and the actual value of the 6 is 6, or 6×1. To the right of the decimal point, the 4 represent that there are 4 $\frac{1}{10}$'s, meaning the actual value of the 4 is $\frac{4}{10}$, or $4 \times \frac{1}{10}$. Contiuing, we see the actual value of the 0 is $\frac{0}{100}$, or $0 \times \frac{1}{100}$, and the actual value of the 3 is $\frac{3}{1000}$, or $3 \times \frac{1}{1000}$. Putting this information together, we get

$$3196.403 = 3 \times 1000 + 1 \times 100 + 9 \times 10 + 6 \times 1 + 4 \times \frac{1}{10} + 0 \times \frac{1}{100} + 3 \times \frac{1}{1000}$$

Now, remembering our powers of ten, we can rewrite this number as

$$3196.403 = 3 \times 10^3 + 1 \times 10^2 + 9 \times 10^1 + 6 \times 10^0 + 4 \times 10^{-1} + 0 \times 10^{-2} + 3 \times 10^{-3}$$

(Recall $a^0 = 1$ and $\frac{1}{a^n} = a^{-n}$ for any number $a \neq 0$.) This is called writing 3196.403 in *expanded notation.*

Example 1 *Write the number* 50237.004 *in expanded notation.*

$$50237.004 = 5 \times 10^4 + 0 \times 10^3 + 2 \times 10^2 + 3 \times 10^1 + 7 \times 10^0 + 0 \times 10^{-1} + 0 \times 10^{-2} + 4 \times 10^{-3}$$

In the above example, since zero times any number is zero, we could shorten the expanded notation above to

$$50237.004 = 5 \times 10^4 + 2 \times 10^2 + 3 \times 10^1 + 7 \times 10^0 + 4 \times 10^{-3}$$

Example 2 *Write the number* 9000403.02001 *in expanded notation.*

$$9000403.02001 = 9 \times 10^6 + 4 \times 10^2 + 3 \times 10^0 + 2 \times 10^{-2} + 1 \times 10^{-5}$$

1.2 Converting from another base to base 10

In the explanation above, there is nothing special about each position representing a power of 10 other than that is what we use. In some instances, it is necessary to use other bases besides 10 for a numeration system. For example, base 2 (binary), base 8 (octal), and base 16 (hexadecimal) systems are very important in computer science. There are also vestiges of numeration systems of different bases in languages around the world. For example, the French word for "eighty" is "quatre-vingt," which means "four twenties." Additionally, there are many different Native American and African languages that emply bases such as three, five, and twelve. So, when a number is expressed in a base other than 10, how can we, as native "base 10" speakers if you will, discern what the value of a number is? We will illustrate this by means of an example.

Example 3 *Express* 4206_8 *as a base 10 (Hindu-Arabic) numeral.*

The subscripted 8 tells us that the given numeral is expressed in base 8 rather than base 10. This means that each position of the numeral represents a power of 8 rather than a power of 10. Just like before, we start at the decimal point and move left. The 6 represents that there are six 1's (because $1 = 8^0$), the 0 represents that there are zero 8's (because $8 = 8^1$), the 2 represents that there are two 64's (because $64 = 8^2$), and the 4 represents that there are four 512's (because $512 = 8^3$). Using expanded notation like we did before, we see

$$4206_8 = 4 \times 8^3 + 2 \times 8^2 + 0 \times 8^1 + 6 \times 8^0$$

Now, we can perform the indicated computations to get

$$\begin{aligned} 4206_8 &= 4 \times 512 + 2 \times 64 + 0 \times 8 + 6 \times 1 \\ &= 2048 + 128 + 0 + 6 \\ &= 2182 \end{aligned}$$

In base 10, $4206_8 = 2182$.

In general, when converting from another base to base 10, we use expanded notation to express the given number, then perform the computations to find the corresponding base 10 numeral.

Example 4 *Express* 10201102_3 *as a base 10 (Hindu-Arabic) numeral.*

$$\begin{aligned} 10201102_3 &= 1 \times 3^7 + 2 \times 3^5 + 1 \times 3^3 + 1 \times 3^2 + 2 \times 3^0 \\ &= 2187 + 486 + 27 + 9 + 2 \\ &= 2711 \end{aligned}$$

So, $10201102_3 = 2711$

1.2.1 How many different digit symbols do we need?

In our base 10 system, we use ten different symbols for the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. We do not need a single digit to represent ten because we have a 10's position; we can represent ten with the two digit symbols 1 and 0. Said another way, we cannot jam both the 1 and the 0 into a single position. This is because the base of our system is 10. We cannot put anything in any single position that is greater than 9. Once we get above 9 for a single position, we put an extra one in the position to the left, and we change the current position to a digit between 0 and 9. (Think about when you add two numbers and what happens when you have to "carry" the one.)

As an example, in a base 8 system, how would we represent "eight?" We cannot simply write "8" because each position represents a power of 8. Just like in base 10 where we cannot put anything in any single position that is greater than 9, in base 8 we cannot put anything in a single position that is greater than 7. Since there is one 8's and no 1's in eight, we put a one in the 8's position and a 0 in the 1's position, thus representing eight as 10_8. Notice that this is exactly the same as representing ten in base 10.

So, in a system of base b, exactly how many different digits do we need? Using our digits for base 10 as our guide, we see that we have ten digits, one for 0 and everything up to 9, which is one less than the base of 10. As we saw above, in base 8, we do not need a digit for 8 because 8 is represented as 10_8. However, we do need a digit for 0 and everything up to 7, which is one less than the base of 8. In general, in a system of base b, we need b different digits, one for each of the numbers 0 up to $b - 1$.

Example 5 *In a positional system of base 7, we need digits for each of the numbers 0, 1, 2, 3, 4, 5, and 6.*

Example 6 *In a positional system of base 12, we need digits for each of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11.*

Notice that in a base 12, we need a *single* digit for the numbers 10 and 11. We cannot use 10 and 11 to represent these numbers because 10_{12} means $1 \times 12^1 + 0 \times 12^0 = 12$ and 11_{12} means $1 \times 12^1 + 1 \times 12^0 = 13$. So, we need to make choices for our digits. Typically, we use T_{12} to represent 10 and E_{12} to represent 11. (We use T_{12} and E_{12} because T is the first letter in ten, and E is the first letter in eleven.)

Example 7 *Convert* $3T4E_{12}$ *to a base 10 (Hindu-Arabic) numeral.*

$$\begin{aligned} 3T4E_{12} &= 3 \times 12^3 + 10 \times 12^2 + 4 \times 12^1 + 11 \times 12^0 \\ &= 5184 + 1440 + 48 + 11 \\ &= 6683 \end{aligned}$$

Example 8 *In a positional system of base 16 (hexadecimal), we need digits for each of the numbers 0, 1, 2, ..., 14, 15. We use* A_{16} *to represent* 10*,* B_{16} *to represent* 11*,* C_{16} *to represent* 12*,* D_{16} *to represent* 13*,* E_{16} *to represent* 14*, and* F_{16} *to represent* 15*.*

Example 9 *Convert* $B31F_{16}$ *to a base 10 (Hindu-Arabic) numeral.*

$$\begin{aligned} B31F_{16} &= 11 \times 16^3 + 3 \times 16^2 + 1 \times 16^1 + 15 \times 16^0 \\ &= 45056 + 768 + 16 + 15 \\ &= 45855 \end{aligned}$$

1.2.2 Converting from base 10 to another base

When converting from another base to base 10, we used expanded notation to rewrite the given numeral, and then we performed the computations to find the corresponding base 10 number. We were able to use expanded notation to rewrite the numeral because we knew the value that each digit represented. However, when we convert from base 10 to another base, we have to figure out what digit to put in each position. Again, we illustrate the process we need to do by example.

Example 10 *Convert* 5402 *to base 7.*

First, we determine what digit to put in the 1's position. In base 7, each position represents a power of 7. In particular, the first position to the left of the decimal point is the 1's position, and each of the other positions are positive powers of 7. This means that the digits in each of the other positions represents some multiple of 7. So, in order to determine the digit that goes in the 1's position, we need to divide the given number by 7; the remainder of this division will be the digit that we put in the 1's position.

$$\begin{array}{rl} 771 & \text{R } 5 \\ 7\,\overline{)5402} & \end{array}$$

Remember that we can find the quotient and remainder using the calculator.

$$\begin{aligned} 5402 \div 7 &= 771.7142857 \\ 771.7142857 - 771 &= 0.714285714 \\ 0.714285714 \times 7 &= 5 \end{aligned}$$

If in the last step you get something like 4.999999999 or 5.000000001, use 5 as the remainder. The slight difference between 5 and these answers represents rounding errors that can occur.

So, 5 will be the digit in the 1's position. Now, we determine what the digit is in the 7's position. The quotient of 771 represents how many times 7 goes into 5402. However, we cannot put 771 into the 7's position. Now, every position to the left of the 7's position is a multiple of 7^2. So, if we divide 771 by 7 and determine the remainder, this remainder will be the digit that goes into the 7's position. This number will represent the part of 5402 that 7 went into once, but not twice.

$$\begin{array}{rl} 110 & \text{R } 1 \\ 7\,\overline{)771} & \text{R } 5 \\ 7\,\overline{)5402} & \end{array}$$

Again, we can find the quotient and remainder using the calculator.

$$\begin{aligned} 771 \div 7 &= 110.1428571 \\ 110.1428571 - 110 &= 0.14285714 \\ 0.14285714 \times 7 &= 1 \end{aligned}$$

So, 1 will be the digit in the 7's position. Now, 110 cannot go in the 7^2's position. So, we divide 110 by 7 again to find the remainder; this remainder will be the digit that goes in the 7^2's position.

$$\begin{array}{rl} 15 & \text{R } 5 \\ 7\,\overline{)110} & \text{R } 1 \\ 7\,\overline{)771} & \text{R } 5 \\ 7\,\overline{)5402} & \end{array}$$

So, 5 will be the digit in the 7^2's position. Again, 15 cannot go in the 7^3's position. So, we divide 15 by 7 to find the remainder; this remainder will be the digit that goes in the 7^3's position.

$$\begin{array}{rl} 2 & \text{R } 1 \\ 7\,\overline{)15} & \text{R } 5 \\ 7\,\overline{)110} & \text{R } 1 \\ 7\,\overline{)771} & \text{R } 5 \\ 7\,\overline{)5402} & \end{array}$$

So, 1 is the digit that goes in the 7^3's position. We see that 2 can go in the 7^4's position. So, this is how we know where to stop. So, $5402 = 21515_7$.

Example 11 *Convert 62053 to base 12.*

In this example, we will divide by 12 successively to determine the digits in base 12.

$$\begin{array}{r}
2 \text{ R } 11 \\
12\overline{)35} \text{ R } 10 \\
12\overline{)430} \text{ R } 11 \\
12\overline{)5171} \text{ R } 1 \\
12\overline{)62053}
\end{array}$$

Again, we know we stop dividing when we get to a quotient that is less than 12. We write the digits in order from last quotient down to first remainder. Remember that $10 = T_{12}$ and $11 = E_{12}$. So, $62053 = 2ETE1_{12}$.

Example 12 *Convert 123456 to base 16.*

$$\begin{array}{r}
1 \text{ R } 14 \\
16\overline{)30} \text{ R } 2 \\
16\overline{)482} \text{ R } 4 \\
16\overline{)7716} \text{ R } 0 \\
16\overline{)123456}
\end{array}$$

Remember that $14 = E_{16}$. So, $123456 = 1E240_{16}$.