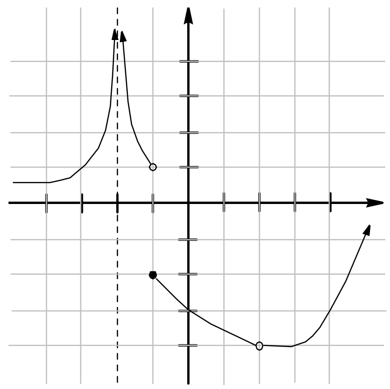
MA 140 - Quiz 1 - Spring 2008
No Calculators Allowed

____, come on down!

1. Answer the following questions concering $y = f(x)$, the graph of which is given below.



Graph of $y = f(x)$

   (a) Evaluate $f(-1)$, if possible. If not, state why.
   $f(-1) = -2$

   (b) Evaluate $\lim_{x \to -1^-} f(x)$, if possible. If the limit does not exist, state why.
   $$\lim_{x \to -1^-} f(x) = 1$$

   (c) Evaluate $\lim_{x \to -1} f(x)$, if possible. If the limit does not exist, state why.
   The limit $\lim_{x \to -1} f(x)$ does not exist because the left-hand and right-hand limits do not agree at $x = -1$.

   (d) Evaluate $\lim_{x \to -2} f(x)$, if possible. If the limit does not exist, state why.
   $$\lim_{x \to -2} f(x) = \infty$$

   (e) Evaluate $\lim_{x \to 0} f(x)$, if possible. If the limit does not exist, state why.
   $$\lim_{x \to 0} f(x) = -3$$

   (f) Evaluate $f(2)$, if possible. If not, state why.
   Since there is a hole at $x = 2$, we see $f(2)$ is not defined.

   (g) Evaluate $\lim_{x \to 2} f(x)$, if possible. If the limit does not exist, state why.
   $$\lim_{x \to 2} f(x) = -4$$

   (h) Find the x-values at which $y = f(x)$ has discontinuities. Classify each discontinuity as either essential or removable.
   This function has essential discontinuities at $x = -2$ and at $x = -1$. The function has a removable discontinuity at $x = 2$.

2. Evaluate the following limits, if they exist. Indicate ∞ or $-\infty$ as appropriate. If a limit does not exist, explain why. Justify your answers.

(a) $\lim_{x\to -2} \frac{3x^2+5x-2}{x^2-x-6}$

$$\begin{aligned}\lim_{x\to -2} \frac{3x^2+5x-2}{x^2-x-6} &= \lim_{x\to -2} \frac{(3x-1)(x+2)}{(x-3)(x+2)} \\ &= \lim_{x\to -2} \frac{3x-1}{x-3} \\ &= \frac{3(-2)-1}{-2-3} \\ &= \frac{7}{5}\end{aligned}$$

(b) $\lim_{x\to 4} \frac{\sqrt{2x+1}-3}{x-4}$

$$\begin{aligned}\lim_{x\to 4} \frac{\sqrt{2x+1}-3}{x-4} &= \lim_{x\to 4} \left(\frac{\sqrt{2x+1}-3}{x-4} \cdot \frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3}\right) \\ &= \lim_{x\to 4} \frac{(2x+1)-9}{(x-4)\left(\sqrt{2x+1}+3\right)} \\ &= \lim_{x\to 4} \frac{2x-8}{(x-4)\left(\sqrt{2x+1}+3\right)} \\ &= \lim_{x\to 4} \frac{2(x-4)}{(x-4)\left(\sqrt{2x+1}+3\right)} \\ &= \lim_{x\to 4} \frac{2}{\sqrt{2x+1}+3} \\ &= \frac{2}{\sqrt{2(4)+1}+3} \\ &= \frac{1}{3}\end{aligned}$$

(c) Using the Squeeze Theorem, evaluate $\lim_{x\to 0} x^6 \cos\left(\frac{1}{x^9}\right)$.

$$\begin{aligned}-1 &\leq \cos\left(\frac{1}{x^9}\right) \leq 1 \\ -x^6 &\leq x^6 \cos\left(\frac{1}{x^9}\right) \leq x^6\end{aligned}$$

Since $\lim_{x\to 0}\left(-x^6\right) = 0 = \lim_{x\to 0} x^6$, we have $\lim_{x\to 0} x^6 \cos\left(\frac{1}{x^9}\right) = 0$ by the Squeeze Theorem.

3. Answer the following.

(a) Let $f(x) = \begin{cases} x^2+2, & x \leq -1 \\ 2x-1, & -1 < x < 3 \\ 8-x, & x > 3 \end{cases}$. Determine if $f(x)$ is continuous at $x=-1$ and at $x=3$. If the function is discontinuous, classify that discontinuity as essential or removable.

At $x=-1$:

1. Is $f(-1)$ defined? Yes, $f(-1) = 3$

2. Does $\lim_{x\to-1} f(x)$ exist?

$$\begin{aligned}
\lim_{x\to-1^-} f(x) &= \lim_{x\to-1^-} \left(x^2+2\right) \\
&= 3 \\
\lim_{x\to-1^+} f(x) &= \lim_{x\to-1^+} (2x-1) \\
&= -3
\end{aligned}$$

Since the one-sided limits do not agree as x approaches -1, we see $\lim_{x\to-1} f(x)$ does not exist.

So, since this limit does not exist, $f(x)$ has an essential discontinuity at $x=-1$.

At $x=3$:

1. Is $f(3)$ defined? No

2. Does $\lim_{x\to3} f(x)$ exist?

$$\begin{aligned}
\lim_{x\to3^-} f(x) &= \lim_{x\to3^-} (2x-1) \\
&= 5 \\
\lim_{x\to3^+} f(x) &= \lim_{x\to3^+} (8-x) \\
&= 5
\end{aligned}$$

Since the one-sided limits agree as x approaches 3, we see $\lim_{x\to-1} f(x) = 5$.

3. Is $\lim_{x\to3} f(x) = f(3)$. No

So, since this limit as x approaches 3 exists but is not equal to the function value, $f(x)$ has a removable discontinuity at $x=3$.

(b) Let $g(x) = \begin{cases} ax^2+ax, & x\leq-2 \\ 3x+2, & x>-2 \end{cases}$. Find the value of a such that $g(x)$ is continuous at $x=-2$. Justify your answer.

In order for $g(x)$ to be continuous at $x=-2$, we need

$$\begin{aligned}
\lim_{x\to-2^-} g(x) &= \lim_{x\to-2^+} g(x) \\
\lim_{x\to-2^-} \left(ax^2+ax\right) &= \lim_{x\to-2^+} (3x+2) \\
4a-2a &= -4 \\
2a &= -4 \\
a &= -2
\end{aligned}$$