MA 140 - Quiz 3 - Spring 2008
No Calculators Allowed

Will the real ____ please stand up?

1. Use the following table of values to answer the questions below.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	4	6	-3	-1
4	2	1	5	2

(a) Let $h(x) = f(\sqrt{x})$. Find $h(4)$ and $h'(4)$.

$$\begin{aligned} h(4) &= f\left(\sqrt{4}\right) \\ &= f(2) \\ &= 4 \end{aligned}$$

$$\begin{aligned} h'(x) &= f'\left(\sqrt{x}\right) \cdot \frac{1}{2}x^{-1/2} \\ h'(4) &= f'\left(\sqrt{4}\right) \cdot \frac{1}{2}(4)^{-1/2} \\ &= f'(2) \cdot \frac{1}{4} \\ &= -\frac{3}{4} \end{aligned}$$

(b) Let $j(x) = g(f(x))$. Find $j(2)$ and $j'(2)$.

$$\begin{aligned} j(2) &= g(f(2)) \\ &= g(4) \\ &= 1 \end{aligned}$$

$$\begin{aligned} j'(x) &= g'(f(x)) \cdot f'(x) \\ j'(2) &= g'(f(2)) \cdot f'(2) \\ &= g'(4) \cdot f'(2) \\ &= 2(-3) \\ &= -6 \end{aligned}$$

(c) Let $k(x) = g(x)h(x)$, where $h(x)$ is as defined in part (a). Find $k(4)$ and $k'(4)$.

$$\begin{aligned} k(4) &= g(4)h(4) \\ &= 1(4) \\ &= 4 \end{aligned}$$

$$\begin{aligned} k'(x) &= g'(x)h(x) + g(x)h'(x) \\ k'(4) &= g'(4)h(4) + g(4)h'(4) \\ &= 2(4) + 1\left(-\frac{3}{4}\right) \\ &= \frac{29}{4} \end{aligned}$$

(d) Let $m(x) = \dfrac{f(x)}{j(x)}$, where $j(x)$ is as defined in part (b). Find $m(2)$ and $m'(2)$.

$$\begin{aligned} m(2) &= \frac{f(2)}{j(2)} \\ &= \frac{4}{1} \\ &= 4 \end{aligned}$$

$$\begin{aligned} m'(x) &= \frac{f'(x)\,j(x) - f(x)\,j'(x)}{[j(x)]^2} \\ m'(2) &= \frac{f'(2)\,j(2) - f(2)\,j'(2)}{[j(2)]^2} \\ &= \frac{-3(1) - 4(-6)}{1^2} \\ &= 21 \end{aligned}$$

2. Let $f(x) = x^7 + 3x - 4$. Then $f^{-1}(x)$ exists. (Take my word for it.)

(a) What is $f^{-1}(-4)$? Justify your answer.
Since $f(0) = -4$, we see $f^{-1}(-4) = 0$

(b) Find $(f^{-1})'(-4)$.
We see $f'(x) = 7x^6 + 3$. Now

$$\begin{aligned} \left(f^{-1}\right)'(-4) &= \frac{1}{f'\left(f^{-1}(-4)\right)} \\ &= \frac{1}{f'(0)} \\ &= \frac{1}{3} \end{aligned}$$

3. Evaluate as indicated.

(a) Find $f'(x)$, if $f(x) = x^{\pi} + \pi^{x} - \sec x - \left(\sqrt{15}\right)^{e-\sqrt{7}}$.

$$f'(x) = \pi x^{\pi-1} + \pi^{x} \ln \pi - \sec x \tan x$$

(b) Find $\dfrac{ds}{dw}$, if $s = \dfrac{\tan\left(w^3 + \ln w\right)}{5 + \csc^2 2w}$.

$$\begin{aligned} \frac{ds}{dw} &= \frac{\sec^2\left(w^3 + \ln w\right) \cdot \left(3w^2 + \frac{1}{w}\right) \cdot \left(5 + \csc^2 2w\right) - \tan\left(w^3 + \ln w\right) \cdot \left(2 \csc 2w \cdot \csc 2w \cot 2w \cdot 2\right)}{\left(5 + \csc^2 2w\right)^2} \\ &= \frac{\left(3w^3 + 1\right)\left(5 + \csc^2 2w\right) \sec^2\left(w^3 + \ln w\right) - 4w \csc^2 2w \cot 2w \tan\left(w^3 + \ln w\right)}{w\left(5 + \csc^2 2w\right)^2} \end{aligned}$$

(c) $\dfrac{d}{dp}\left(r(p)\right)$, if $r(p) = \sin\left(\ln\left(\cos p\right)\right)$

$$\begin{aligned} r'(p) &= \cos\left(\ln\left(\cos p\right)\right) \cdot \frac{1}{\cos p} \cdot \left(-\sin p\right) \\ &= -\cos\left(\ln\left(\cos p\right)\right) \tan p \end{aligned}$$

(d) y', if $y = e^{\sqrt{x}} \ln(x^2+2x+1)$

$$\begin{aligned} y' &= e^{\sqrt{x}} \cdot \left(-\frac{1}{2}x^{-1/2}\right) \cdot \ln\left(x^2+2x+1\right) + e^{\sqrt{x}} \cdot \frac{1}{x^2+2x+1} \cdot (2x+2) \\ &= -\frac{e^{\sqrt{x}} \ln(x^2+2x+1)}{2\sqrt{x}} + \frac{e^{\sqrt{x}}(2x+2)}{x^2+2x+1} \end{aligned}$$

(e) $j'(x)$, if $j(x) = (\cot x)^{x^3}$.

$$\begin{aligned} j(x) &= (\cot x)^{x^3} \\ \ln j(x) &= \ln (\cot x)^{x^3} \\ \ln j(x) &= x^3 \ln(\cot x) \\ \frac{1}{j(x)} \cdot j'(x) &= 3x^2 \ln(\cot x) + x^3 \cdot \frac{1}{\cot x} \cdot \left(-\csc^2 x\right) \\ j'(x) &= j(x)\left(3x^2 \ln(\cot x) - \frac{x^3 \csc^2 x}{\cot x}\right) \\ &= (\cot x)^{x^3}\left(3x^2 \ln(\cot x) - \frac{x^3 \csc^2 x}{\cot x}\right) \end{aligned}$$

4. Using the definition of the derivative, prove the Product Rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

$$\begin{aligned} (f(x)g(x))' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h}\right) \\ &= \lim_{h\to 0} \left(\frac{(f(x+h) - f(x))g(x+h)}{h} + \frac{f(x)(g(x+h) - g(x))}{h}\right) \\ &= \lim_{h\to 0} \left(\frac{(f(x+h) - f(x))}{h} \cdot g(x+h)\right) + \lim_{h\to 0}\left(f(x) \cdot \frac{(g(x+h) - g(x))}{h}\right) \\ &= \lim_{h\to 0} \frac{(f(x+h) - f(x))}{h} \cdot \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} \frac{(g(x+h) - g(x))}{h} \\ &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$