Gram-Schmidt Practice

1. Let $\mathbb{R}^3$ have the usual Euclidean inner product.

   (a) Show $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

   $$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Since the rref of this matrix is the identity matrix, we conclude that the original set of vectors is a basis for $\mathbb{R}^3$.

   (b) Explain why this is not an orthonormal basis for $\mathbb{R}^3$.

   This is not an orthonormal basis because (1) the vectors are not pairwise orthogonal (the second and third are not orthogonal), and (2) the vectors do not all have magnitude 1.

   (c) Using the Gram-Schmidt process, transform this basis into an orthonormal basis for $\mathbb{R}^3$.

   $$\overrightarrow{w_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   $$\begin{aligned} \overrightarrow{w_2} &= \overrightarrow{v_2} - \frac{\overrightarrow{w_1} \cdot \overrightarrow{v_2}}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}} \overrightarrow{w_1} \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{0}{3} \overrightarrow{w_1} \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

   $$\begin{aligned} \overrightarrow{w_3} &= \overrightarrow{v_3} - \frac{\overrightarrow{w_1} \cdot \overrightarrow{v_3}}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}} \overrightarrow{w_1} - \frac{\overrightarrow{w_2} \cdot \overrightarrow{v_3}}{\overrightarrow{w_2} \cdot \overrightarrow{w_2}} \overrightarrow{w_2} \\ &= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \end{bmatrix} \end{aligned}$$

An orthogonal basis is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \end{bmatrix} \right\}$$

Normalizing, we find the orthonormal basis

$$\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{\frac{1}{6}}} \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \end{bmatrix} \right\}$$

2. Let $\mathbb{R}^4$ have the usual Euclidean inner product.

(a) Show $\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^4$.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the rref of this matrix is the identity matrix, we conclude that this set of vectors is a basis for $\mathbb{R}^4$.

(b) Using the Gram-Schmidt process, transform this basis into an orthonormal basis for $\mathbb{R}^4$.

$$\overrightarrow{w_1} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \overrightarrow{w_2} &= \overrightarrow{v_2} - \frac{\overrightarrow{w_1} \cdot \overrightarrow{v_2}}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}} \overrightarrow{w_1} \\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{-2}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -\frac{1}{5} \\ \frac{2}{5} \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
\overrightarrow{w_3} &= \overrightarrow{v_3} - \frac{\overrightarrow{w_1}\cdot\overrightarrow{v_3}}{\overrightarrow{w_1}\cdot\overrightarrow{w_1}}\overrightarrow{w_1} - \frac{\overrightarrow{w_2}\cdot\overrightarrow{v_3}}{\overrightarrow{w_2}\cdot\overrightarrow{w_2}}\overrightarrow{w_2} \\
&= \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{\frac{3}{5}}{\frac{6}{5}}\begin{bmatrix} 1 \\ -\frac{1}{5} \\ \frac{2}{5} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ -1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{w_4} &= \overrightarrow{v_4} - \frac{\overrightarrow{w_1}\cdot\overrightarrow{v_4}}{\overrightarrow{w_1}\cdot\overrightarrow{w_1}}\overrightarrow{w_1} - \frac{\overrightarrow{w_2}\cdot\overrightarrow{v_4}}{\overrightarrow{w_2}\cdot\overrightarrow{w_2}}\overrightarrow{w_2} - \frac{\overrightarrow{w_3}\cdot\overrightarrow{v_4}}{\overrightarrow{w_3}\cdot\overrightarrow{w_3}}\overrightarrow{w_3} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{0}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{\frac{6}{5}}\begin{bmatrix} 1 \\ -\frac{1}{5} \\ \frac{2}{5} \\ 0 \end{bmatrix} - \frac{-\frac{1}{2}}{\frac{5}{2}}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{15} \\ \frac{4}{15} \\ -\frac{8}{15} \\ \frac{4}{5} \end{bmatrix}
\end{aligned}$$

So, an orthogonal basis is

$$\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{1}{5} \\ \frac{2}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{4}{15} \\ \frac{4}{15} \\ -\frac{8}{15} \\ \frac{4}{5} \end{bmatrix} \right\}$$

Normalizing, we get

$$\left\{ \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{\frac{6}{5}}}\begin{bmatrix} 1 \\ -\frac{1}{5} \\ \frac{2}{5} \\ 0 \end{bmatrix}, \frac{1}{\sqrt{\frac{5}{2}}}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{\frac{16}{15}}}\begin{bmatrix} \frac{4}{15} \\ \frac{4}{15} \\ -\frac{8}{15} \\ \frac{4}{5} \end{bmatrix} \right\}$$

3. Let $\mathbb{R}^3$ have the inner product $\overrightarrow{v}\cdot\overrightarrow{w} = v_1w_1 + 2v_2w_2 + 3v_3w_3$.

(a) Verify $\cdot$ is in fact an inner product on $\mathbb{R}^3$.

Let $\overrightarrow{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\overrightarrow{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ and $\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(1) We have $\overrightarrow{v}\cdot\overrightarrow{v} = v_1^2+2v_2^2+3v_3^2$. Since $v_1^2, v_2^2, v_3^2 \geq 0$, we have $v_1^2+2v_2^2+3v_3^2 \geq 0$. So, $\overrightarrow{v}\cdot\overrightarrow{v} \geq 0$ for all $\overrightarrow{v} \in \mathbb{R}^3$. Further, we see $\overrightarrow{v}\cdot\overrightarrow{v} = 0$ if and only if $v_1 = v_2 = v_3 = 0$, or if and only if $\overrightarrow{v} = \overrightarrow{0}$.

(2)

$$\begin{aligned}
\overrightarrow{v}\cdot\overrightarrow{w} &= v_1w_1+2v_2w_2+3v_3w_3\\
&= w_1v_1+2w_2v_2+3w_3v_3\\
&= \overrightarrow{w}\cdot\overrightarrow{v}
\end{aligned}$$

(3)

$$\begin{aligned}
\overrightarrow{v}\cdot\left(\overrightarrow{w}+\overrightarrow{x}\right) &= \begin{bmatrix} v_1\\ v_2\\ v_3\end{bmatrix}\cdot\left(\begin{bmatrix} w_1\\ w_2\\ w_3\end{bmatrix}+\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}\right)\\
&= \begin{bmatrix} v_1\\ v_2\\ v_3\end{bmatrix}\cdot\begin{bmatrix} w_1+x_1\\ w_2+x_2\\ w_3+x_3\end{bmatrix}\\
&= v_1\left(w_1+x_1\right)+2v_2\left(w_2+x_2\right)+3v_3\left(w_3+x_3\right)\\
&= v_1w_1+v_1x_1+2v_2w_2+2v_2x_2+3v_3w_3+3v_3x_3\\
&= v_1w_1+2v_2x_2+3v_3w_3+v_1x_1+2v_2w_2+3v_3x_3\\
&= \begin{bmatrix} v_1\\ v_2\\ v_3\end{bmatrix}\cdot\begin{bmatrix} w_1\\ w_2\\ w_3\end{bmatrix}+\begin{bmatrix} v_1\\ v_2\\ v_3\end{bmatrix}\cdot\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}\\
&= \overrightarrow{v}\cdot\overrightarrow{w}+\overrightarrow{v}\cdot\overrightarrow{x}
\end{aligned}$$

(4)

$$\begin{aligned}
\alpha\left(\overrightarrow{v}\cdot\overrightarrow{w}\right) &= \alpha\left(v_1w_1+2v_2w_2+3v_3w_3\right)\\
&= \alpha v_1w_1+2\alpha v_2w_2+3\alpha v_3w_3\\
&= \left(\alpha\overrightarrow{v}\right)\cdot\overrightarrow{w}\\
&= v_1\alpha w_1+2v_2\alpha w_2+3v_3\alpha w_3\\
&= \overrightarrow{v}\cdot\left(\alpha\overrightarrow{w}\right)
\end{aligned}$$

So, $\cdot$ is an inner product on $\mathbb{R}^3$.

(b) Use the Gram-Schmidt process to transform the basis $\left\{\begin{bmatrix} 1\\ 1\\ 1\end{bmatrix},\begin{bmatrix} 1\\ 1\\ 0\end{bmatrix},\begin{bmatrix} 1\\ 0\\ 0\end{bmatrix}\right\}$ into an orthonormal basis for $\mathbb{R}^3$.

$$\overrightarrow{w_1}=\begin{bmatrix} 1\\ 1\\ 1\end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} 1\\ 1\\ 0\end{bmatrix}\cdot\begin{bmatrix} 1\\ 1\\ 1\end{bmatrix} &= 1\left(1\right)+2\left(1\right)\left(1\right)+3\left(0\right)\left(1\right)\\
&= 3
\end{aligned}$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} = 1(1)+2(1)(1)+3(1)(1) = 6$$

$$\begin{aligned}\overrightarrow{w_2} &= \overrightarrow{v_2} - \frac{\overrightarrow{w_1}\cdot\overrightarrow{v_2}}{\overrightarrow{w_1}\cdot\overrightarrow{w_1}}\overrightarrow{w_1}\\ &= \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{3}{6}\begin{bmatrix}1\\1\\1\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} = 1(1)+2(0)(1)+3(0)(1) = 1$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} = 1\left(\frac{1}{2}\right)+2(0)\left(\frac{1}{2}\right)+3(0)\left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+3\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{3}{2}$$

$$\begin{aligned}\overrightarrow{w_3} &= \overrightarrow{v_3} - \frac{\overrightarrow{w_1}\cdot\overrightarrow{v_3}}{\overrightarrow{w_1}\cdot\overrightarrow{w_1}}\overrightarrow{w_1} - \frac{\overrightarrow{w_2}\cdot\overrightarrow{v_3}}{\overrightarrow{w_2}\cdot\overrightarrow{w_2}}\overrightarrow{w_2}\\ &= \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{6}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{\frac{1}{2}}{\frac{3}{2}}\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}\\ &= \begin{bmatrix}\frac{2}{3}\\-\frac{1}{3}\\0\end{bmatrix}\end{aligned}$$

So, an orthogonal basis with respect to this inner product is

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix},\begin{bmatrix}\frac{2}{3}\\-\frac{1}{3}\\0\end{bmatrix}\right\}$$

Normalizing with respect to this basis, we get an orthonormal basis

$$\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{\frac{3}{2}}} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \frac{1}{\sqrt{\frac{2}{3}}} \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix} \right\}$$

4. Let P_2 have the inner product $f \cdot g = \int_0^1 fg\ dx$. Transform the standard basis $\{1, x, x^2\}$ into an orthonormal basis for P_2.

$$\overrightarrow{w_1} = 1$$

$$\begin{aligned} \overrightarrow{w_2} &= \overrightarrow{v_2} - \frac{\overrightarrow{w_1} \cdot \overrightarrow{v_2}}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}} \overrightarrow{w_1} \\ &= x - \frac{\int_0^1 1x\ dx}{\int_0^1 1^2\ dx} (1) \\ &= x - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \overrightarrow{w_3} &= \overrightarrow{v_3} - \frac{\overrightarrow{w_1} \cdot \overrightarrow{v_3}}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}} \overrightarrow{w_1} - \frac{\overrightarrow{w_2} \cdot \overrightarrow{v_3}}{\overrightarrow{w_2} \cdot \overrightarrow{w_2}} \overrightarrow{w_2} \\ &= x^2 - \frac{\int_0^1 1x^2\ dx}{\int_0^1 1^2\ dx} (1) - \frac{\int_0^1 \left(x - \frac{1}{2}\right) x^2\ dx}{\int_0^1 \left(x - \frac{1}{2}\right)^2\ dx} \left(x - \frac{1}{2}\right) \\ &= x^2 - x + \frac{1}{6} \end{aligned}$$

So, an orthogonal basis is

$$\left\{ 1, x - \frac{1}{2}, x^2 - x + \frac{1}{6} \right\}$$

Normalizing, we get

$$\begin{aligned} &\left\{ \frac{1}{\sqrt{\int_0^1 1\ dx}} (1), \frac{1}{\sqrt{\int_0^1 \left(x - \frac{1}{2}\right)^2\ dx}} \left(x - \frac{1}{2}\right), \frac{1}{\sqrt{\int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2\ dx}} \left(x^2 - x + \frac{1}{6}\right) \right\} \\ = &\left\{ 1, \frac{1}{\sqrt{\frac{1}{12}}} \left(x - \frac{1}{2}\right), \frac{1}{\sqrt{\frac{1}{180}}} \left(x^2 - x + \frac{1}{6}\right) \right\} \end{aligned}$$