MA 303 - Linear Algebra - Exam 1 - Spring 2008

Directions: Answer the following questions on the paper provided. Please begin each new problem on a separate sheet of paper and only write on one side of the paper. Show all your work. An answer with no work receives NO credit. You may use your calculator to check your answers, but simply stating, "I did the problem on my calculator" does not constitute work.

1. (1 point each) Answer the following with either **TRUE** or **FALSE**.

   (a) FALSE - For all matrices A, B, and C such that AB and AC are defined, if $AB = AC$, then $B = C$.

   (b) FALSE - If A and B are nonsingular matrices such that $A + B$ is defined, then $A + B$ is also nonsingular.

   (c) FALSE - If the number of rows of A is strictly less than the number of columns of A, the system $A\overrightarrow{x} = \overrightarrow{b}$ is consistent.

   (d) FALSE - If the number of rows of A is strictly greater than the number of columns of A, the system $A\overrightarrow{x} = \overrightarrow{b}$ is inconsistent.

   (e) TRUE - If A and B are nonsingular matrices so that $AB = BA$, then $(AB)^{-1} = A^{-1}B^{-1}$.

   (f) TRUE - The matrix
   $$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -6 & 12 \end{bmatrix}$$
   is in reduced row-echelon form.

   (g) FALSE - A matrix with a column of zeros has an inverse.

   (h) TRUE - Suppose the reduced row echelon form of a matrix A has a column without a pivot. Then the system $A\overrightarrow{x} = \overrightarrow{0}$ will have infinitely many solutions.

   (i) FALSE - If $AB = 0$, then either A or B has at least one entry that is zero.

   (j) FALSE - We have class tomorrow morning.

2. Let $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$, $\quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$,

   $D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, $\quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$.

   Compute the following, if possible. If the operation is not possible, state why.

   (a) (2 points) $2B - C$

   This operation is not possible because $2B$ is a 2×2 matrix and C is a 2×3 matrix.

   (b) (2 points) BC

$$\begin{aligned} BC &= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4(1) + (-1)3 & 4(4) + (-1)1 & 4(2) + (-1)5 \\ 0(1) + 2(3) & 0(4) + 2(1) & 0(2) + 2(5) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix} \end{aligned}$$

(c) (3 points) tr(CA)

$$\begin{aligned} CA &= \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(3)+4(-1)+2(1) & 1(0)+4(2)+2(1) \\ 3(3)+1(-1)+5(1) & 3(0)+1(2)+5(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 10 \\ 13 & 7 \end{bmatrix} \\ \operatorname{tr}(CA) &= 1+7 \\ &= 8 \end{aligned}$$

(d) (3 points) $EC^T - 4A$

$$\begin{aligned} EC^T - 4A &= \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T - 4 \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} - 4 \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6(1)+1(4)+3(2) & 6(3)+1(1)+3(5) \\ (-1)(1)+1(4)+2(2) & (-1)(3)+1(1)+2(5) \\ 4(1)+1(4)+3(2) & 4(3)+1(1)+3(5) \end{bmatrix} - \begin{bmatrix} 4(3) & 4(0) \\ 4(-1) & 4(2) \\ 4(1) & 4(1) \end{bmatrix} \\ &= \begin{bmatrix} 16 & 34 \\ 7 & 8 \\ 14 & 28 \end{bmatrix} - \begin{bmatrix} 12 & 0 \\ -4 & 8 \\ 4 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 34 \\ 11 & 0 \\ 10 & 24 \end{bmatrix} \end{aligned}$$

3. (10 points) Find the inverse of $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$ by hand, or explain why the inverse does not exist.

$$\left[\begin{array}{cccccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_1+R_3 \to R_3} \left[\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & -1 & 1 \end{array}\right]$$

$$\xrightarrow{2R_3 \to R_3} \left[\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & -2 & 2 \end{array}\right]$$

$$\xrightarrow[-2R_3+R_2 \to R_2]{\frac{1}{2}R_3+R_1 \to R_1} \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 6 & 5 & -4 \\ 0 & 0 & 1 & -3 & -2 & 2 \end{array}\right]$$

So,

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 6 & 5 & -4 \\ -3 & -2 & 2 \end{bmatrix}$$

4. (10 points) Find conditions on a, b, and c so that the system

$$\begin{array}{rcrcrcl} 2x_1 & + & 3x_2 & - & x_3 & = & a \\ x_1 & - & x_2 & + & 3x_3 & = & b \\ 3x_1 & + & 7x_2 & - & 5x_3 & = & c \end{array}$$

is consistent. You do not need to solve the system.

$$\begin{bmatrix} 2 & 3 & -1 & a \\ 1 & -1 & 3 & b \\ 3 & 7 & -5 & c \end{bmatrix} \xrightarrow{R_1 \sim R_2} \begin{bmatrix} 1 & -1 & 3 & b \\ 2 & 3 & -1 & a \\ 3 & 7 & -5 & c \end{bmatrix}$$

$$\xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & -1 & 3 & b \\ 0 & 5 & -7 & a-2b \\ 0 & 10 & -14 & -3b+c \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & -1 & 3 & b \\ 0 & 5 & -7 & a-2b \\ 0 & 0 & 0 & -2a+b+c \end{bmatrix}$$

If this system is to be consistent, we need the bottom row to be a row of zeros. So, we need $-2a+b+c=0$, or $b+c=2a$.

5. (10 points) Solve the following system.

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 4 \\ 2x_1 & - & 3x_2 & + & 2x_3 & - & 3x_4 & = & -1 \\ 3x_1 & - & 5x_2 & + & 3x_3 & - & 4x_4 & = & 3 \\ -x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 5 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & -1 & 4 \\ 2 & -3 & 2 & -3 & -1 \\ 3 & -5 & 3 & -4 & 3 \\ -1 & 1 & -1 & 2 & 5 \end{bmatrix} \xrightarrow[\substack{-3R_1+R_3 \to R_3 \\ R_1+R_4 \to R_4}]{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & -2 & 1 & -1 & 4 \\ 0 & 1 & 0 & -1 & -9 \\ 0 & 1 & 0 & -1 & -9 \\ 0 & -1 & 0 & 1 & 9 \end{bmatrix}$$

$$\xrightarrow[\substack{-R_2+R_3 \to R_3 \\ R_2+R_4 \to R_4}]{2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 1 & -3 & -14 \\ 0 & 1 & 0 & -1 & -9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that x_3 and x_4 are free variables; let $x_3 = s$ and $x_4 = t$. Then $x_1 = -14 - s + 3t$ and $x_2 = -9 + t$. So, the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -14 - s + 3t \\ -9 + t \\ s \\ t \end{bmatrix}$$

$$= \begin{bmatrix} -14 \\ -9 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

6. We call λ an *eigenvalue* of A if $(A-\lambda I)\overrightarrow{x}=\overrightarrow{0}$ has nontrivial solutions. (Note that the matrix $A-\lambda I$ is simply the matrix A with λ subtracted from the diagonal elements.) We call a vector $\overrightarrow{v}\neq\overrightarrow{0}$ an *eigenvector* of a matrix A associated with λ if $\overrightarrow{v}$ is a solution of $(A-\lambda I)\overrightarrow{x}=\overrightarrow{0}$.

(a) (10 points) For the matrix $A=\begin{bmatrix}-1 & -6\\ 1 & 4\end{bmatrix}$, find the eigenvalues of A by determining the values of λ so that the system $(A-\lambda I)\overrightarrow{x}=\overrightarrow{0}$ has nontrivial solutions.

$$\begin{bmatrix}-1-\lambda & -6 & 0\\ 1 & 4-\lambda & 0\end{bmatrix} \xrightarrow{R_1\sim R_2} \begin{bmatrix}1 & -\lambda+4 & 0\\ -\lambda-1 & -6 & 0\end{bmatrix}$$

$$\xrightarrow{(1+\lambda)R_1+R_2\to R_2} \begin{bmatrix}1 & -\lambda+4 & 0\\ 0 & (\lambda+1)(-\lambda+4)-6 & 0\end{bmatrix}$$

If this system is to have infinitely many solutions, we need

$$\begin{aligned}(\lambda+1)(-\lambda+4)-6 &= 0\\ -\lambda^2+3\lambda+2 &= 0\\ -(\lambda-1)(\lambda-2) &= 0\\ \lambda &= 1,2\end{aligned}$$

(b) (10 points) For each eigenvalue λ you found in part (a), find all eigenvectors associated with λ by substituting the value you found for λ and solving the system.

For $\lambda=1$, we have

$$\begin{bmatrix}1 & 3 & 0\\ 0 & 0 & 0\end{bmatrix}$$

We see x_2 is a free variable; let $x_2=t$. Then $x_1=-3t$. So, the solutions are

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=t\begin{bmatrix}-3\\ 1\end{bmatrix}$$

For $\lambda=2$, we have

$$\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

We see x_2 is a free variable; let $x_2=t$. Then $x_1=-2t$. So, the solutions are

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=t\begin{bmatrix}-2\\ 1\end{bmatrix}$$

7. Recall that the conjugate transpose of A, denoted A^*, is $\left(\overline{A}\right)^T$, the matrix formed by taking the complex conjugate of each entry of A and then taking the transpose. A square matrix A composed of real number entries is called *orthogonal* if $AA^T=I$ (or equivalently, $A^T=A^{-1}$). A square matrix A composed of complex number entries is called *unitary* if $AA^*=I$ (or equivalently, $A^*=A^{-1}$.)

(a) (5 points) Show that $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix for all $\theta \in \mathbb{R}$.

$$\begin{aligned}
AA^T &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \backslash \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta(\cos\theta) + (-\sin\theta)(-\sin\theta) & \cos\theta(\sin\theta) + (-\sin\theta)(\cos\theta) \\ \sin\theta(\cos\theta) + (\cos\theta)(-\sin\theta) & \sin\theta(\sin\theta) + (\cos\theta)(\cos\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

So, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix for all $\theta \in \mathbb{R}$.

(b) (5 points) Show that $B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}i \end{bmatrix}$ is a unitary matrix. (Recall that $i = \sqrt{-1}$.)

$$\begin{aligned}
BB^T &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \end{bmatrix}^* \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) & \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}i\right) + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}i\right) \\ -\frac{1}{\sqrt{2}}i\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}i\left(\frac{1}{\sqrt{2}}\right) & -\frac{1}{\sqrt{2}}i\left(\frac{1}{\sqrt{2}}i\right) + \frac{1}{\sqrt{2}}i\left(-\frac{1}{\sqrt{2}}i\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & -\frac{1}{2}i^2 + \frac{1}{2}i^2 \\ -\frac{1}{2}i + \frac{1}{2}i & -\frac{1}{2}i^2 - \frac{1}{2}i^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

(c) (5 points) Prove that if P and Q are orthogonal matrices, then PQ is an orthogonal matrix. We are given that $PP^T = I$ and $QQ^T = I$. So, we have

$$\begin{aligned}
(PQ)(PQ)^T &= (PQ)\left(Q^T P^T\right) \\
&= P\left(QQ^T\right)P^T \\
&= P(I)P^T \\
&= PP^T \\
&= I
\end{aligned}$$

So, PQ is an orthogonal matrix.

8. Let $C = [c_{ij}]$ be an $m \times n$ matrix.

(a) (5 points) Determine the general entry of CC^T in terms of the entries of C. In other words, if $CC^T = [d_{ij}]$, find a general formula for d_{ij}.

Let $C^T = \left[c'_{ij}\right]$, where $c'_{ij} = c_{ji}$. Then

$$\begin{aligned} CC^T &= \left[c_{ij}\right]\left[c'_{ij}\right] \\ &= \left[\sum_{k=1}^{n} c_{ik}c'_{kj}\right] \\ &= \left[\sum_{k=1}^{n} c_{ik}c_{jk}\right] \end{aligned}$$

So, $d_{ij} = \sum_{k=1}^{n} c_{ik}c_{jk}$

(b) (10 points) Recall the trace of an $m \times m$ matrix $D = [d_{ij}]$, denoted $\text{tr}(D)$, is the sum of the diagonal entries of D, i.e.

$$\text{tr}\,(D) = \sum_{i=1}^{m} d_{ii}$$

Prove $\text{tr}\left(CC^T\right) = s$, where s is the sum of the squares of all the entries of C.

$$\begin{aligned} \text{tr}\left(CC^T\right) &= \text{tr}\left(\left[\sum_{k=1}^{n} c_{ik}c_{jk}\right]\right) \\ &= \sum_{l=1}^{m}\sum_{k=1}^{n} c_{lk}c_{lk} \\ &= \sum_{l=1}^{m}\sum_{k=1}^{n} \left(c_{lk}\right)^2 \end{aligned}$$

We see that this double sum runs through all the rows and all the columns of C. So, $\text{tr}\left(CC^T\right)$ is equal to the sum of the squares of all the entries of C.