MA 303 - Exam 1 Review - Spring 2008

1. The *nullspace* of a matrix A is defined as the set of solutions of the system $A\overrightarrow{x} = \overrightarrow{0}$.

   (a) Find the null space of the matrix $A = \begin{bmatrix} 1 & 2 & -2 & 0 & 1 \\ 2 & 4 & -1 & 0 & -4 \\ -3 & -6 & 12 & 2 & -12 \\ 1 & 2 & -2 & -4 & -5 \end{bmatrix}$.

   Adjoining the zero vector and row reducing, we obtain

$$\begin{bmatrix} 1 & 2 & -2 & 0 & 1 & 0 \\ 2 & 4 & -1 & 0 & -4 & 0 \\ -3 & -6 & 12 & 2 & -12 & 0 \\ 1 & 2 & -2 & -4 & -5 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   The free variables are x_2 and x_5; let $x_2 = s$ and $x_5 = t$. Then $x_1 = -2s + 3t$, $x_3 = 2t$, and $x_4 = -\frac{3}{2}t$. So, the set of solutions to the system $A\overrightarrow{x} = \overrightarrow{0}$, and hence the null space of A, is

$$\begin{bmatrix} -2s+3t \\ s \\ 2t \\ -\frac{3}{2}t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 2 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

   (b) The vector $\mathbf{v} = \begin{bmatrix} -12 \\ 0 \\ -5 \\ 8 \\ 0 \end{bmatrix}$ is a solution to the system

$$\begin{bmatrix} 1 & 2 & -2 & 0 & 1 \\ 2 & 4 & -1 & 0 & -4 \\ -3 & -6 & 12 & 2 & -12 \\ 1 & 2 & -2 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ -19 \\ -8 \\ -34 \end{bmatrix}$$

   (You do NOT need to show this.) Write the set of all solutions of $A\overrightarrow{x} = \overrightarrow{b}$, where $\overrightarrow{b} = \begin{bmatrix} -2 \\ -19 \\ -8 \\ -34 \end{bmatrix}$

   Since $s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 2 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$ is the set of solutions to $A\overrightarrow{x} = \overrightarrow{0}$, the set of solutions to $A\overrightarrow{x} = \overrightarrow{b}$ will be

$$\begin{bmatrix} -12 \\ 0 \\ -5 \\ 8 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 2 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

2. Answer the following:

   (a) Determine if there exist scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that $\alpha_1 \overrightarrow{v}_1 + \alpha_2 \overrightarrow{v}_2 + \alpha_3 \overrightarrow{v}_3 + \alpha_4 \overrightarrow{v}_4 = \overrightarrow{b}$, where

$$\overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \overrightarrow{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -2 \\ 5 \end{bmatrix}, \overrightarrow{v}_3 = \begin{bmatrix} -3 \\ -6 \\ 1 \\ -8 \end{bmatrix}, \overrightarrow{v}_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -4 \end{bmatrix}, \overrightarrow{b} = \begin{bmatrix} 8 \\ 17 \\ -8 \\ 3 \end{bmatrix}.$$

   We need to solve the system of equations

$$\begin{aligned} \alpha_1 + 2\alpha_2 - 3\alpha_3 &= 8 \\ 2\alpha_1 + 5\alpha_2 - 6\alpha_3 &= 17 \\ -\alpha_1 - 2\alpha_2 + \alpha_3 - \alpha_4 &= -8 \\ 5\alpha_2 - 8\alpha_3 - 4\alpha_4 &= 3 \end{aligned}$$

   Using the augmented matrix and row reducing, we see

$$\begin{bmatrix} 1 & 2 & -3 & 0 & 8 \\ 2 & 5 & -6 & 0 & 17 \\ -1 & -2 & 1 & -1 & -8 \\ 0 & 5 & -8 & -4 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

   The last row implies $0 = 1$, which is false. Hence, the system is inconsistent, and there do not exist scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that $\alpha_1 \overrightarrow{v}_1 + \alpha_2 \overrightarrow{v}_2 + \alpha_3 \overrightarrow{v}_3 + \alpha_4 \overrightarrow{v}_4 = \overrightarrow{b}$.

   (b) A set of vectors $\{\overrightarrow{v}_1, \overrightarrow{v}_2, ... \overrightarrow{v}_n\}$ in a vector space V is said to be *linearly independent* if $k_1 \overrightarrow{v}_1 + k_2 \overrightarrow{v}_2 + \cdots + k_n \overrightarrow{v}_n = \overrightarrow{0}$ implies $k_1 = k_2 = \cdots = k_n = 0$. For example, the vectors $\overrightarrow{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\overrightarrow{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent in $\mathbb{R}^2$ because if $k_1 \overrightarrow{e_1} + k_2 \overrightarrow{e}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we get by equating entries that

$$\begin{aligned} 1k_1 + 0k_2 &= 0 \\ 0k_1 + 1k_2 &= 0 \end{aligned}$$

   and the only solution to this system is $k_1 = k_2 = 0$.

Determine if the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in part (a) are linearly independent by finding a system of equations, writing the associated augmented matrix, row-reducing, and determining if the zero vector is the ONLY solution to the system.

We need to solve the system

$$\begin{array}{rcrcrcrcr} k_1 & + & 2k_2 & - & 3k_3 & & & = & 0 \\ 2k_1 & + & 5k_2 & - & 6k_3 & & & = & 0 \\ -k_1 & - & 2k_2 & + & k_3 & - & k_4 & = & 0 \\ & & 5k_2 & - & 8k_3 & - & 4k_4 & = & 0 \end{array}$$

Using an augmented matrix and row reducing, we see

$$\begin{bmatrix} 1 & 2 & -3 & 0 & 0 \\ 2 & 5 & -6 & 0 & 0 \\ -1 & -2 & 1 & -1 & 0 \\ 0 & 5 & -8 & -4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a free variable in the system, the system will have infinitely many solutions. So, the zero vector is NOT the only solution to the system, meaning the vectors are not linearly independent.

(c) Determine if the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

are linearly independent by finding a system of equations, writing the associated augmented matrix, row-reducing, and determining if the zero vector is the ONLY solution to the system.

We need to solve the system

$$\begin{array}{rcrcrcr} k_1 & + & 5k_2 & + & 3k_3 & = & 0 \\ -2k_1 & + & 6k_2 & + & 2k_3 & = & 0 \\ 3k_1 & - & k_2 & + & k_3 & = & 0 \end{array}$$

Using an augmented matrix and row reducing, we see

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a free variable in the system, the system will have infinitely many solutions. So, the zero vector is NOT the only solution to the system, meaning the vectors are not linearly independent.

3. Solve the following system, or explain why the following system has no solution.

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 4 \\ -x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 5 \\ 2x_1 & - & 3x_2 & + & 2x_3 & - & 3x_4 & = & -1 \\ & & x_2 & & & - & x_4 & = & -9 \end{array}$$

Using an augmented matrix and row reducing

$$\begin{bmatrix} 1 & -2 & 1 & -1 & 4 \\ -1 & 1 & -1 & 2 & 5 \\ 2 & -3 & 2 & -3 & -1 \\ 0 & 1 & 0 & -1 & -9 \end{bmatrix} \xrightarrow{\substack{R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3}} \begin{bmatrix} 1 & -2 & 1 & -1 & 4 \\ 0 & -1 & 0 & 1 & 9 \\ 0 & 1 & 0 & -1 & -9 \\ 0 & 1 & 0 & -1 & -9 \end{bmatrix}$$

$$\xrightarrow{-R_2\to R_2} \begin{bmatrix} 1 & -2 & 1 & -1 & 4 \\ 0 & 1 & 0 & -1 & -9 \\ 0 & 1 & 0 & -1 & -9 \\ 0 & 1 & 0 & -1 & -9 \end{bmatrix}$$

$$\xrightarrow{\substack{2R_2+R_1\to R_1 \\ -R_2+R_3\to R_3 \\ -R_2+R_4\to R_4}} \begin{bmatrix} 1 & 0 & 1 & -3 & -14 \\ 0 & 1 & 0 & -1 & -9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The variables x_3 and x_4 are free variables. Let $x_3 = s$ and $x_4 = t$. Then $x_1 = -14 - s + 3t$ and $x_2 = -9 + t$. So, the set of solutions is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} -14 - s + 3t \\ -9 + t \\ s \\ t \end{bmatrix} \\ &= \begin{bmatrix} -14 \\ -9 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

4. Find conditions on b_1, b_2, b_3 so that

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$$

is consistent.

Using an augmented matrix and row reducing, we see

$$\begin{bmatrix} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{bmatrix} \xrightarrow{\substack{-4R_1+R_2\to R_2 \\ 3R_1+R_3\to R_3}} \begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1+b_2 \\ 0 & -3 & 12 & 3b_1+b_3 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3\to R_3} \begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1+b_2 \\ 0 & 0 & 0 & -b_1+b_2+b_3 \end{bmatrix}$$

So, in order for this system to be consistent, we must have $-b_1 + b_2 + b_3 = 0$, or $b_1 = b_2 + b_3$.

5. Find (if possible) the inverses of the following matrices. If the matrix has no inverse, explain why it does not.

   (a) $A = \begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

$$\begin{bmatrix} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_1 \to R_1} \begin{bmatrix} 1 & -3 & 4 & -1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{-2R_1+R_2 \to R_2 \\ 4R_1+R_3 \to R_3}} \begin{bmatrix} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{10}R_2 \to R_2} \begin{bmatrix} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 1 & -\frac{7}{10} & \frac{1}{5} & \frac{1}{10} & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{3R_2+R_1 \to R_1 \\ 10R_2+R_3 \to R_3}} \begin{bmatrix} 1 & 0 & \frac{19}{10} & -\frac{2}{5} & \frac{3}{10} & 0 \\ 0 & 1 & -\frac{7}{10} & \frac{1}{5} & \frac{1}{10} & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{bmatrix}$$

There is no way now to get a pivot in the third column. In other words, the original matrix is row equivalent to a matrix with a row of zeros. Hence, A is not invertible.

(b) $B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \sim R_2} \begin{bmatrix} 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[-2R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 7 & 2 & 1 & -2 & 0 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{7}R_2 \to R_2} \begin{bmatrix} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow[-4R_2+R_3 \to R_3]{2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -\frac{3}{7} & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} & -\frac{6}{7} & 1 \end{bmatrix}$$

$$\xrightarrow{-7R_3 \to R_3} \begin{bmatrix} 1 & 0 & -\frac{3}{7} & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{bmatrix}$$

$$\xrightarrow[-\frac{2}{7}R_3+R_2 \to R_2]{\frac{3}{7}R_3+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{bmatrix}$$

So, $B^{-1} = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$.

6. Consider the system

$$\begin{aligned} (a-\lambda)x + \quad by &= 0 \\ cx + (d-\lambda)y &= 0 \end{aligned}$$

(a) Can this system ever be inconsistent? Why or why not?

This is a homogeneous system, and the zero vector is always a solution to a homogeneous system. So, this system can never be inconsistent.

(b) Show that the values of λ for which the system has nontrivial solutions should satisfy the quadratic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

You may assume that $a - \lambda$ and c are not equal to zero.

Using an augmented matrix and row reducing, we see

$$\begin{bmatrix} a-\lambda & b & 0 \\ c & d-\lambda & 0 \end{bmatrix} \xrightarrow{\frac{1}{a-\lambda}R_1 \to R_1} \begin{bmatrix} 1 & \frac{b}{a-\lambda} & 0 \\ c & d-\lambda & 0 \end{bmatrix}$$

$$\xrightarrow{-cR_1+R_2 \to R_2} \begin{bmatrix} 1 & \frac{b}{a-\lambda} & 0 \\ 0 & d-\lambda-\frac{bc}{a-\lambda} & 0 \end{bmatrix}$$

In order for this system to have nontrivial solutions, we must have a free variable. Thus, we need to have no pivot in the second column. To get this, we must have

$$\begin{aligned} d-\lambda-\frac{bc}{a-\lambda} &= 0 \\ (a-\lambda)\left(d-\lambda-\frac{bc}{a-\lambda}\right) &= 0\,(a-\lambda) \\ (a-\lambda)(d-\lambda)-bc &= 0 \\ ad-a\lambda-d\lambda+\lambda^2-bc &= 0 \\ \lambda^2-(a+d)\lambda+(ad-bc) &= 0 \end{aligned}$$

7. Let

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Perform the following operations, if possible. If the operation is not possible, state why.

(a) A^2

Since A is not a square matrix, this operation is not possible.

(b) $(3C^T - A)B$

$$\begin{aligned}
C^T &= \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} \\
3C^T &= \begin{bmatrix} 3 & 9 \\ 12 & 3 \\ 6 & 15 \end{bmatrix} \\
3C^T - A &= \begin{bmatrix} 3 & 9 \\ 12 & 3 \\ 6 & 15 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 9 \\ 13 & 1 \\ 5 & 14 \end{bmatrix} \\
\left(3C^T - A\right) B &= \begin{bmatrix} 0 & 9 \\ 13 & 1 \\ 5 & 14 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 18 \\ 52 & -11 \\ 20 & 23 \end{bmatrix}
\end{aligned}$$

(c) D^2

$$\begin{aligned}
D^2 &= \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 9 & 15 \\ 2 & -3 & 2 \\ 13 & 23 & 24 \end{bmatrix}
\end{aligned}$$

(d) $EA + C^T$

$$EA = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 5 \\ -2 & 4 \\ 14 & 5 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}$$

$$EA + C^T = \begin{bmatrix} 20 & 5 \\ -2 & 4 \\ 14 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 8 \\ 2 & 5 \\ 16 & 10 \end{bmatrix}$$

8. Consider the matrix

$$A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$$

(a) Using elementary row-operations, find a matrix B so that A is row-equivalent to B, and B is in row-echelon form.

$$\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 1 & 3 & 3 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 2 & 7 & 7 \end{bmatrix}$$

$$\xrightarrow{-2R_1 + R_3 \to R_3} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{-R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix is now in row-echelon form.

(b) Is A row-equivalent to the identity matrix? Why or why not?

Performing two more elementary row operations

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\xrightarrow{-3R_3+R_1\to R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since have gone from A to the identity matrix by using only elementary row operations, we see that A is row-equivalent to the identity matrix.

(c) If A is row-equivalent to the identity matrix, find a sequence E_1, E_2, ..., E_n of elementary matrices so that $E_nE_{n-1}\cdots E_1 = A^{-1}$. (You do NOT need to multiply the matrices together.) If not, explain why A^{-1} does not exist.

Writing elementary matrices, in order, that realize the row operations, we have:

$$E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$
$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$
$$E_5 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$E_6 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $A^{-1} = E_6E_5E_4E_3E_2E_1$.

9. Consider the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

(a) Write the system of equations in the form $Ax = b$.

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$$

(b) Write the system of equations as an augmented matrix $[A \mid b]$.

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

(c) Using the augmented matrix and row operations, solve the system.

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$$\xrightarrow{-3R_1+R_3 \to R_3} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$\xrightarrow{-R_2 \to R_2} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$\xrightarrow{10R_2+R_3 \to R_3} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$$

$$\xrightarrow{-52R_3 \to R_3} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

So, $z = 2$. Using this value, we have

$$\begin{aligned} y - 5(2) &= -9 \\ y &= 1 \end{aligned}$$

Using these two values, we have

$$\begin{aligned} x + 1 + 2(2) &= 8 \\ x &= 3 \end{aligned}$$

10. Show that the matrix

$$X = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

is singular for any values of the entries a, b, c, d, e, f, g, h. (HINT: Show X is row-equivalent to a matrix with a row of zeros and then explain why X must be singular.)

If a is 0 or h is 0, then X has a row of zeros and thus is singular. If not, we can do the following:

$$\begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix} \xrightarrow{R_1 \sim R_2} \begin{bmatrix} b & 0 & c & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{a}R_2 \to R_2} \begin{bmatrix} b & 0 & c & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

$$\xrightarrow{-dR_2 + R_3 \to R_3} \begin{bmatrix} b & 0 & c & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

Now, if e is 0, we have that the original matrix is row-equivalent to a matrix with a row of zeros. If e is not 0, we can do the following:

$$\begin{bmatrix} b & 0 & c & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix} \xrightarrow{R_3 \sim R_4} \begin{bmatrix} b & 0 & c & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{e}R_4 \to R_4} \begin{bmatrix} b & 0 & c & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

$$\xrightarrow{-hR_4 + R_5 \to R_5} \begin{bmatrix} b & 0 & c & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In any case, either X has a row of zeros, or X is row-equivalent to a matrix with a row of zeros. If X is row-equivalent to a matrix with a row of zeros, there is no way for X to be row-equivalent to the identity matrix, which is necessary for X to be nonsingular. Thus, X must be singular.

11. A square matrix A is called *symmetric* if $A^T = A$.

    (a) For any matrix A, prove that $A^T A$ is symmetric.

    To prove $A^T A$ is symmetric, we must show that $\left(A^T A\right)^T = A^T A$.

    $$\begin{aligned}\left(A^T A\right)^T &= A^T \left(A^T\right)^T \\ &= A^T A\end{aligned}$$

    Hence, $A^T A$ is symmetric.

    (b) Suppose A is a square matrix such that $A^T A = A$. Use part (a) to prove that A is symmetric.

    Since $A^T A$ is symmetric and $A = A^T A$, it must be the case that A is symmetric.

    (c) Use part (b) to show if $A = A^T A$, then A is *idempotent*; that is, prove that if $A = A^T A$, then $A^2 = A$.

    $$\begin{aligned}A^2 &= AA \\ &= A^T A \text{ since } A = A^T \\ &= A \text{ by the given}\end{aligned}$$

12. A matrix $A = [a_{ij}]$ is *lower triangular* if all the entries above the main diagonal are 0. Notationally, we have $a_{ij} = 0$ whenever $i < j$. Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are $n \times n$ upper triangular matrices. By appealing to the elements of these matrices, prove that AB is also a lower triangular matrix; in other words, if $AB = [c_{ij}]$, show $c_{ij} = 0$ if $i < j$.

    Let $A = [a_{ij}]$ and $B = [b_{ij}]$ with $a_{ij} = b_{ij} = 0$ whenever $i < j$. If $i < j$, then the i, j-entry of AB is:

    $$\begin{aligned}\sum_{k=1}^{n} a_{ik} b_{kj} &= a_{i1} b_{ij} + a_{i2} b_{2j} + \cdots + a_{i(i-1)} b_{(i-1)j} + a_{ii} b_{ij} \\ &\quad + a_{i(i+1)} b_{(i+1)j} + \cdots + a_{in} b_{nj} \\ &= a_{i1} \cdot 0 + a_{i2} \cdot 0 + \cdots + a_{i(i-1)} \cdot 0 + a_{ii} \cdot 0 \\ &\quad + 0 \cdot b_{(i+1)j} + \cdots + 0 \cdot b_{nj} \\ &= 0\end{aligned}$$

    Since the i, j-entry of AB is 0, the matrix is lower triangular.

13. For a square matrix A, prove that $\left(A^k\right)^T = \left(A^T\right)^k$ for all integers $k \geq 1$.

    By induction on k:

    Basis step ($k = 1$): $\left(A^1\right)^T = A^T = \left(A^T\right)^1$.

Inductive step: Suppose $\left(A^k\right)^T = \left(A^T\right)^k$ is true for some value of k. For $k+1$:

$$\begin{aligned}
\left(A^{k+1}\right)^T &= \left(A^k A\right)^T \\
&= A^T \left(A^k\right)^T \\
&= A^T \left(A^T\right)^k \text{ by the inductive hypothesis} \\
&= \left(A^T\right)^{k+1}
\end{aligned}$$

Hence, by induction, $\left(A^k\right)^T = \left(A^T\right)^k$ for all integers $k \geq 1$.