# MATH 341 - Linear Algebra - Exam 2 Review - Spring 2008

1. Answer the following with either **TRUE** or **FALSE**.

   (a) The span of a set of five distinct nonzero vectors from $\mathbb{R}^4$ will always equal all of $\mathbb{R}^4$.

   **FALSE** - The span of the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is clearly not all of $\mathbb{R}^4$ because the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is not in the span of these vectors.

   (b) The span of a set of three distinct nonzero vectors from $\mathbb{R}^4$ can never equal all of $\mathbb{R}^4$.

   **TRUE**

   (c) If $\text{rank}(A) < \text{rank}\left(\left[A \mid \vec{b}\right]\right)$, then the system $A\vec{x} = \vec{b}$ is inconsistent.

   **TRUE**

   (d) Suppose $A$ is an $m \times n$ matrix. The set of solutions to the system $A\vec{x} = \vec{0}$ forms a subspace of $\mathbb{R}^m$.

   **FALSE** - The set of solutions to this system forms a subspace of $\mathbb{R}^n$.

   (e) If $W_1$ and $W_2$ are subspaces of a vector space $V$, then $W_1 \cup W_2$ is never a subspace of $V$.

   **FALSE** - If either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then $W_1 \cup W_2$ is a subspace of $V$.

   (f) If $A$ is invertible and $A$ and $B$ are multiplicatively compatible, then $\text{rank}(AB) = \text{rank}(B)$.

   **TRUE**

   (g) If $\dim(V) = 5$ and $S$ has five elements, then $S$ is a basis for $V$.

   **FALSE -** The set $S$ would need to span $V$ (or be linearly independent).

   (h) If $\dim(V) = 7$ and $S$ has nine elemnents, then $S$ is linearly dependent.

   **TRUE**

   (i) If $S$ spans $V$, then every vector in $V$ can be written uniquely as a linear combination of vectors in $S$.

   **FALSE** - $S$ must be linearly independent and span $V$ in order for this to be true.

   (j) If $\dim(V) = 8$ and $S$ has 10 elements, then $S$ can be reduced to form a basis for $V$.

   **FALSE** - This is true only if $S$ also spans $V$.

   (k) If $\vec{0} \in S$, then $S$ is linearly dependent.

   **TRUE**

   (l) If $A$ is an $m \times n$ matrix, then the row space and column space of $A$ are a subspaces of $\mathbb{R}^n$.

   **FALSE** - The row space is a subspace of $\mathbb{R}^n$, but the column space is a subspace of $\mathbb{R}^m$.

   (m) If $A$ is an $m \times n$ matrix, then the dimension of the column space of $A$ plus the dimension of the nullspace of $A$ equals $m$.

   **FALSE** - The number of pivots in rref (i.e. the dimension of the column space) plus the number of free variables in rref (i.e. the dimension of the nullspace) equals the number of *columns*, not the number of rows.

2. Let

$$B = \begin{bmatrix} 1 & 2 & 0 & 3 & 2 & 5 \\ -1 & 1 & -3 & 3 & 4 & 7 \\ 0 & -2 & 2 & -4 & 1 & -3 \\ 2 & 0 & 4 & -2 & 0 & -2 \\ 1 & 0 & 2 & -1 & 1 & 0 \end{bmatrix}$$

See problems 17 and 18.

3. A couple questions about bases and subspaces...justify your answers!

(a) If $V$ is a vector space of dimension 8, if $U$ is a subspace of $V$ of dimension 5, and if $W$ is a subspace of dimension 6, what are the minimum and maximum dimensions $U \cap W$ can have?

Since $U \cap W$ is a subspace of both $U$ and $W$, the largest dimension $U \cap W$ can have is the smaller of the dimensions of $U$ and $W$. So, the maximum dimension $U \cap W$ can have is 5.

Recall that $U + W = \{\vec{u} + \vec{w} \mid \vec{u} \in U, \vec{w} \in W\}$. Let $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ be a basis for $U \cap W$. Since $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ is a linearly independent subset of vectors of $U$ (each vector in the set is in both $U$ and $W$), we can add vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_m$ from $U$ to $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ so that the set $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{u}_1, \vec{u}_2, ..., \vec{u}_m\}$ is a basis for $U$. Similarly, since $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ is a linearly independent subset of vectors of $W$, we can add vectors $\vec{w}_1, \vec{w}_2, ..., \vec{w}_n$ from $W$ to $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ so that the set $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{w}_1, \vec{w}_2, ..., \vec{w}_n\}$ is a basis for $W$.

Now, consider the set $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{u}_1, \vec{u}_2, ..., \vec{u}_m, \vec{w}_1, \vec{w}_2, ..., \vec{w}_n\}$. We claim that this set is a basis for $U + W$. To show that this set spans $U + W$, let $\vec{u} + \vec{w} \in U + W$. Since $\vec{u}$ is in $U$, we can write $\vec{u}$ as a linear combination of $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{u}_1, \vec{u}_2, ..., \vec{u}_m$. Since $\vec{w}$ is in $W$, we can write $\vec{w}$ as a linear combination of $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{w}_1, \vec{w}_2, ..., \vec{w}_n$. So, we can write $\vec{u} + \vec{w}$ as a linear combination of $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{u}_1, \vec{u}_2, ..., \vec{u}_m, \vec{w}_1, \vec{w}_2, ..., \vec{w}_n$. Hence, $\vec{u} + \vec{w}$ is in the span of $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{u}_1, \vec{u}_2, ..., \vec{u}_m, \vec{w}_1, \vec{w}_2, ..., \vec{w}_n\}$.

To show that this set is linearly independent, we know that the set $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{u}_1, \vec{u}_2, ..., \vec{u}_m\}$ is linearly independent by construction (it is a basis for $U$). If $\vec{w}_i$ is in the span of $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{u}_1, \vec{u}$ for some $i = 1, ..., n$, we would be able to express $\vec{w}_i$ as a linear combination of vectors that all live in $U$. Since $U$ is a subspace, any linear combination of vectors from $U$ is in $U$, which would make $\vec{w}_i$ live in $U$. By construction, $\vec{w}_i$ is in $W$; hence, $\vec{w}_i \in U \cap W$. Since $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ is a basis for $U \cap W$, we now have that $\vec{w}_i$ is a linear combination of $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$, making $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{w}_i\}$ a linearly dependent set. This contradicts the fact that $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{w}_1, \vec{w}_2, ..., \vec{w}_n\}$ is a basis for $W$. Thus, $\vec{w}_i$ is not in the span of $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{u}_1, \vec{u}_2, ..., \vec{u}_m\}$ for all $i = 1, ..., n$. Applying this $n$ times, we get $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{u}_1,$ is linearly independent.

Since $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{u}_1, \vec{u}_2, ..., \vec{u}_m, \vec{w}_1, \vec{w}_2, ..., \vec{w}_n\}$ is linearly independent and spans $U + W$, we get $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k, \vec{u}_1, \vec{u}_2, ..., \vec{u}_m, \vec{w}_1, \vec{w}_2, ..., \vec{w}_n\}$ is a basis for $U + W$. So,

$$\begin{aligned} \dim(U + W) &= k + m + n \\ &= k + m + k + n - k \\ &= \dim(U) + \dim(W) - \dim(U \cap W) \end{aligned}$$

Since $U + W$ is a subspace of $V$, the largest dimension it can have is 8. So, we have

$$\begin{aligned} 8 &\geq 5 + 6 - \dim(U \cap W) \\ \dim(U \cap W) &\geq 3 \end{aligned}$$

(b) Suppose $S$ is a linearly independent set of 6 vectors in $\mathbb{R}^{15}$. What is the maximum number of vectors that could be added to $S$ and have the resulting set still be linearly independent?

Since the dimension of $\mathbb{R}^{15}$ is 15, the maximum number of elements of we can add and still have a linearly independent set is 9.

(c) Suppose $S$ is a set of 17 vectors in $M_{43}$, the set of all $4 \times 3$ matrices with real number entries. What is the minimum number of vectors that could be removed from $S$ to form a linearly independent set? What requirement must $S$ satisfy to obtain this minimum number? (HINT: Think basis.)

The dimension of $M_{43}$ is 12. So, we must remove at least 5 elements from the set to form a linearly independent set. If we only need to remove 5 vectors, then this set will be a basis. This means the original set must have been a spanning set for $M_{43}$.

4. Answer the following.

(a) The set of all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is a subspace of $M_{22}$. (You do NOT need to prove this.) Find a basis for this subspace and determine the dimension of this subspace.

Let $\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. So, $\{\vec{v}_1, \vec{v}_2\}$ spans this subspace. Clearly, $\vec{v}_2$ is not a nonzero scalar multiple of $\vec{v}_1$; hence, $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. This gives $\{\vec{v}_1, \vec{v}_2\}$ is a basis for this subspace, and hence its dimension is 2.

(b) Show that $\vec{p}_1 = 1 + x$, $\vec{p}_2 = 1 + x^2$, $\vec{p}_3 = x + x^2$ is a basis for $P_2$, and find the coordinates of $\vec{p} = 2 - x + x^2$ with respect to this basis.

We can answer both questions by looking for constants $k_1, k_2, k_3$ so that $k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3 = \vec{p}$. By equating like terms, we get the system of equations

$$\begin{aligned} k_1 + k_2 &= 2 \\ k_1 + k_3 &= -1 \\ k_2 + k_3 &= 1 \end{aligned}$$

Setting up an augmented matrix and row reducing, we get

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Since the coefficient matrix is row equivalent to the identity matrix, we have that $S = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is a linearly independent set. Finally, $\vec{p} = (0, 2, -1)_S$.

(c) Let $\vec{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 7 \\ -8 \\ 1 \end{bmatrix}$. Find a basis for the subspace of $\mathbb{R}^3$ spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ that uses only vectors from this set, and express the vectors not in the basis as a linear combination of the basis vectors.

To find a basis for the subspace these vectors span, we put the vectors as columns in a matrix, row reduce, and use the vectors that are the columns in the original matrix that correspon to the columns in the matrix in rref that have pivots.

$$\begin{bmatrix} 3 & 1 & 6 & 7 \\ -2 & 1 & -4 & -8 \\ 1 & 1 & 2 & 1 \end{bmatrix} \overset{R}{\sim} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first two columns have pivots; hence, a basis for the subspace is $\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Also, from the rref, we see that $2 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix}$ and $3 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \\ 1 \end{bmatrix}$.

5. Let $\{\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3\}$ be a basis for a vector space $V$. Show that $\{\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3\}$ is also a basis, where $\overrightarrow{u}_1 = \overrightarrow{v}_1$, $\overrightarrow{u}_2 = \overrightarrow{v}_1 - \overrightarrow{v}_2$, $\overrightarrow{u}_3 = 2\overrightarrow{v}_1 - 2\overrightarrow{v}_2 + \overrightarrow{v}_3$.

   Since $\{\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3\}$ is a basis for a vector space $V$, every basis must have three elements. So, since $\{\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3\}$ has three elements, we only need to show that this set is linearly independent. To this end, suppose $k_1\overrightarrow{u}_1 + k_2\overrightarrow{u}_2 + k_3\overrightarrow{u}_3 = \overrightarrow{0}$. We then have

$$\begin{aligned} k_1\overrightarrow{u}_1 + k_2\overrightarrow{u}_2 + k_3\overrightarrow{u}_3 &= \overrightarrow{0} \\ k_1\overrightarrow{v}_1 + k_2\left(\overrightarrow{v}_1 - \overrightarrow{v}_2\right) + k_3\left(2\overrightarrow{v}_1 - 2\overrightarrow{v}_2 + \overrightarrow{v}_3\right) &= \overrightarrow{0} \\ (k_1 + k_2 + 2k_3)\,\overrightarrow{v}_1 + (-k_2 - 2k_3)\,\overrightarrow{v}_2 + k_3\overrightarrow{v}_3 &= \overrightarrow{0} \end{aligned}$$

   Since $\{\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3\}$ is a basis, we know that

$$\begin{aligned} k_1 + k_2 + 2k_3 &= 0 \\ -k_2 - 2k_3 &= 0 \\ k_3 &= 0 \end{aligned}$$

   Solving this system, we see

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \overset{R}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

   So, if $k_1\overrightarrow{u}_1 + k_2\overrightarrow{u}_2 + k_3\overrightarrow{u}_3 = \overrightarrow{0}$, then $k_1 = k_2 = k_3 = 0$. This gives $\{\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3\}$ is linearly independent and hence a basis for $V$.

6. The following questions refer to vector spaces.

   (a) Show that $\mathbb{R}^2$ with addition defined as $(x, y) \oplus (r, s) = (x + r + 1, y + s)$ and scalar multiplication defined as $\alpha \odot (x, y) = (\alpha x + \alpha - 1, \alpha y)$ is a vector space. (I am not trying to trick you here. This IS a vector space.)

   1.

$$\begin{aligned} (x, y) + (r, s) &= (x + r + 1, y + s) \\ &\in \mathbb{R}^2 \end{aligned}$$

   So, axiom 1 holds.

   2.

$$\begin{aligned} (x, y) + (r, s) &= (x + r + 1, y + s) \\ &= (r + x + 1, s + y) \\ &= (r, s) + (x, y) \end{aligned}$$

   So, axiom 2 holds.

3.

$$\begin{aligned}(x,y)+((r,s)+(t,u)) &= (x,y)+(r+t+1,s+u)\\ &= (x+(r+t+1)+1,y+(s+u))\\ &= ((x+r+1)+t+1,(y+s)+u)\\ &= (x+r+1,y+s)+(t+u)\\ &= ((x,y)+(r,s))+(t,u)\end{aligned}$$

So, axiom 3 holds.

4. Let $\mathbf{0}=(-1,0)$. Then,

$$\begin{aligned}(x,y)+\mathbf{0} &= (x,y)+(-1,0)\\ &= (x-1+1,y+0)\\ &= (x,y)\end{aligned}$$

and

$$\begin{aligned}\mathbf{0}+(x,y) &= (-1,0)+(x,y)\\ &= (-1+x+1,0+y)\\ &= (x,y)\end{aligned}$$

So, axiom 4 holds.

5. Let $-(x,y)=(2-x,-y)$. Then,

$$\begin{aligned}(x,y)+(2-x,-y) &= (x+(2-x)+1,y-y)\\ &= (-1,0)\\ &= \mathbf{0}\end{aligned}$$

So, axiom 5 holds.

6.

$$\begin{aligned}\alpha(x,y) &= (\alpha x+\alpha-1,\alpha y)\\ &\in \mathbb{R}^2\end{aligned}$$

So, axiom 6 holds.

7.

$$\begin{aligned}\alpha((x,y)+(r,s)) &= \alpha(x+r+1,y+s)\\ &= (\alpha(x+r+1)+\alpha-1,\alpha(y+s))\\ &= (\alpha x+\alpha r+\alpha+\alpha-1,\alpha y+\alpha s)\\ &= (\alpha x+\alpha-1+\alpha r+\alpha-1+1,\alpha y+\alpha s)\\ &= (\alpha x+\alpha-1,\alpha y)+(\alpha r+\alpha-1,\alpha s)\\ &= \alpha(x,y)+\alpha(r,s)\end{aligned}$$

So, axiom 7 holds.

8.

$$\begin{aligned}(\alpha+\beta)(x,y) &= ((\alpha+\beta)x+(\alpha+\beta)-1,(\alpha+\beta)y) \\ &= (\alpha x+\alpha-1+\beta x+\beta-1+1,\alpha y+\beta y) \\ &= (\alpha x+\alpha-1,\alpha y)+(\beta x+\beta-1,\beta y) \\ &= \alpha(x,y)+\beta(x,y)\end{aligned}$$

So, axiom 8 holds.

9.

$$\begin{aligned}\alpha(\beta(x,y)) &= \alpha(\beta x+\beta-1,\beta y) \\ &= (\alpha(\beta x+\beta-1)+\alpha-1,\alpha(\beta y)) \\ &= ((\alpha\beta)x+(\alpha\beta)-1,(\alpha\beta)y) \\ &= (\alpha\beta)(x,y)\end{aligned}$$

So, axiom 9 holds.

10.

$$\begin{aligned}1(x,y) &= (x+1-1,y) \\ &= (x,y)\end{aligned}$$

So, axiom 10 holds.

Since all 10 axioms hold, this is a vector space.

(b) Determine whether or not $\mathbb{R}^2$ with ordinary scalar multiplication, but scalar multiplication defined as $k\odot(x,y)=(0,0)$, where $k\in\mathbb{R}$, is a vector space.

Checking axiom (S4), we see

$$\begin{aligned}1\odot(x,y) &= (0,0) \\ &\neq (x,y)\end{aligned}$$

So, (S4) fails, and thus this is not a vector space.

(c) Prove that, in any vector space $V$, we have $0\overrightarrow{v}=\overrightarrow{0}$, where $\overrightarrow{0}$ is the zero vector of $V$, and $\overrightarrow{v}$ is any vector in $V$.

$$\begin{aligned}0\overrightarrow{v} &= (0+0)\overrightarrow{v} \\ 0\overrightarrow{v} &= 0\overrightarrow{v}+0\overrightarrow{v} \text{ by axiom S2} \\ -(0\overrightarrow{v})+0\overrightarrow{v} &= -(0\overrightarrow{v})+(0\overrightarrow{v}+0\overrightarrow{v}) \text{ by axiom A3} \\ \overrightarrow{0} &= -(0\overrightarrow{v})+(0\overrightarrow{v}+0\overrightarrow{v}) \text{ by axiom A3} \\ \overrightarrow{0} &= (-(0\overrightarrow{v})+0\overrightarrow{v})+0\overrightarrow{v} \text{ by axiom A1} \\ \overrightarrow{0} &= \overrightarrow{0}+0\overrightarrow{v} \text{ by axiom A3} \\ \overrightarrow{0} &= 0\overrightarrow{v} \text{ by axiom A2}\end{aligned}$$

7. The following questions refer to subspaces.

(a) The set $V$ of all everywhere differentiable functions (i.e. $f'(x)$ exists for all $x\in\mathbb{R}$) forms a vector space under usual addition and scalar multiplication. (You do NOT need to check this fact.) Let $W=\{f(x)\in V \mid f'(2)=0\}$. Determine if $W$ is a subspace of $V$.

Let $f(x), g(x) \in W$. Then,

$$\begin{aligned}(f+g)'(2) &= f'(2)+g'(2)\\ &= 0+0\\ &= 0\end{aligned}$$

Hence, $(f+g)(x) \in W$. Also,

$$\begin{aligned}(\alpha f)'(2) &= \alpha f'(2)\\ &= \alpha \cdot 0\\ &= 0\end{aligned}$$

Hence, $(\alpha f)(x) \in W$. Hence, $W$ is a subspace of $V$.

(b) The set $C[a,b]$ of all continuous functions on the interval $[a,b]$ forms a vector space under usual addition and scalar multiplication. (You do NOT need to check this fact.) Determine if

$$W = \left\{ f \in C[a,b] \mid \int_a^b f\,dx = 1 \right\}$$

is a subspace of $C[a,b]$.

If $f, g \in W$, then $\int_a^b f\,dx = 1 = \int_a^b g\,dx = 1$. We have

$$\begin{aligned}\int_a^b (f+g)\ dx &= \int_a^b f\,dx + \int_a^b g\,dx\\ &= 1+1\\ &= 2\end{aligned}$$

So, $f+g \notin W$, and $W$ is not a subspace of $V$.

(c) Let $U$ and $W$ be subspaces of a vector space $V$. Show that $U \cap W$ is also a subspace of $V$ by showing the closure axioms hold.

Let $\mathbf{u}, \mathbf{w} \in U \cap W$. Then, $\mathbf{u}, \mathbf{w} \in U$ and $\mathbf{u}, \mathbf{w} \in W$. Since $U$ and $W$ are subspaces of $V$, we have $\mathbf{u}+\mathbf{w} \in U$ and $\mathbf{u}+\mathbf{w} \in W$. Hence, $\mathbf{u}+\mathbf{w} \in U \cap W$. Also, $\alpha\mathbf{u} \in U$ and $\alpha\mathbf{u} \in W$. So, $\alpha\mathbf{u} \in U \cap W$, and $U \cap W$ is a subspace of $V$.

8. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

(a) Determine if $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$.

(b) Let $\mathbf{u}_1 = \mathbf{v}_1$. Compute $\mathbf{u}_2 = \mathbf{v}_2 - \operatorname{proj}_{\mathbf{u}_1}\mathbf{v}_2$.

(c) Compute $\mathbf{u}_3 = \mathbf{v}_3 - \operatorname{proj}_{\mathbf{u}_1}\mathbf{v}_3 - \operatorname{proj}_{\mathbf{u}_2}\mathbf{v}_3$.

(d) Compute $\mathbf{u}_1 \cdot \mathbf{u}_2$, $\mathbf{u}_1 \cdot \mathbf{u}_3$, and $\mathbf{u}_2 \cdot \mathbf{u}_3$.

(e) Determine unit vectors in the same direction as $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

a. Consider

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ 1 & -1 & -4 \end{bmatrix}$$

Row reducing, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This means the column space of this matrix is all of $\mathbb{R}^3$. Hence, span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$.

b.

$$\begin{aligned} \text{proj}_{\mathbf{u}_1}\mathbf{v}_2 &= \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \\ &= \frac{-2-3-1}{1+1+1}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{u}_2 &= \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_2 \\ &= \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

c.

$$\begin{aligned} \text{proj}_{\mathbf{u}_1}\mathbf{v}_3 &= \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \\ &= \frac{1-2-4}{1+1+1}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{5}{3} \\ \frac{5}{3} \\ -\frac{5}{3} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{proj}_{\mathbf{u}_2}\mathbf{v}_3 &= \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \\ &= \frac{0+2-4}{0+1+1}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{u}_3 &= \mathbf{v}_3 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_3 - \text{proj}_{\mathbf{u}_2}\mathbf{v}_3 \\ &= \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} - \begin{bmatrix} -\frac{5}{3} \\ \frac{5}{3} \\ -\frac{5}{3} \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{8}{3} \\ \frac{4}{3} \\ -\frac{4}{3} \end{bmatrix} \end{aligned}$$

d.

$$\begin{aligned}\mathbf{u}_1 \cdot \mathbf{u}_2 &= 0 - 1 + 1 \\ &= 0\end{aligned}$$

$$\begin{aligned}\mathbf{u}_1 \cdot \mathbf{u}_3 &= \frac{8}{3} - \frac{4}{3} - \frac{4}{3} \\ &= 0\end{aligned}$$

$$\begin{aligned}\mathbf{u}_2 \cdot \mathbf{u}_3 &= 0 + \frac{4}{3} - \frac{4}{3} \\ &= 0\end{aligned}$$

e.

$$\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{3}{4\sqrt{6}} \begin{bmatrix} \frac{8}{3} \\ \frac{4}{3} \\ -\frac{4}{3} \end{bmatrix}$$

9. In a vector space, prove $k\mathbf{0} = \mathbf{0}$ for all scalars $k$.

$$\begin{aligned}k\mathbf{0} &= k(\mathbf{0} + \mathbf{0}) \\ k\mathbf{0} &= k\mathbf{0} + k\mathbf{0} \\ k\mathbf{0} + (-k\mathbf{0}) &= (k\mathbf{0} + k\mathbf{0}) + (-k\mathbf{0}) \\ \mathbf{0} &= k\mathbf{0} + (k\mathbf{0} + (-k\mathbf{0})) \\ \mathbf{0} &= k\mathbf{0} + \mathbf{0} \\ \mathbf{0} &= k\mathbf{0}\end{aligned}$$

10. Let $V = \{f \mid f(x) > 0 \text{ for all } x \in \mathbb{R}\}$ with addition defined as

$$(f+g)(x) = f(x) \cdot g(x)$$

and scalar multiplication defined as

$$(\alpha f)(x) = [f(x)]^{\alpha}$$

Determine if $V$ is a vector space.

(1) Certainly, $(f+g)(x) = f(x) \cdot g(x) \in V$ because $f(x)$ and $g(x)$ are positive, making their product positive.

(2)

$$\begin{aligned}(f+g)(x) &= f(x) \cdot g(x) \\ &= g(x) \cdot f(x) \\ &= (g+f)(x)\end{aligned}$$

(3)

$$\begin{aligned}(f+(g+h))(x) &= (f+gh)(x)\\ &= f(x)(g(x)h(x))\\ &= (f(x)g(x))h(x)\\ &= (f+g)(x)h(x)\\ &= ((f+g)+h)(x)\end{aligned}$$

(4) Let $0(x)=1$. Then

$$\begin{aligned}(f+0)(x) &= f(x)0(x)\\ &= f(x)\\ &= 0(x)f(x)\\ &= (0+f)(x)\end{aligned}$$

(5) Let $-f(x)=\frac{1}{f(x)}$. Since $f(x)>0$ for all $x$, $-f(x)\in V$. Then

$$\begin{aligned}(f+(-f))(x) &= f(x)\frac{1}{f(x)}\\ &= 1\\ &= 0(x)\end{aligned}$$

and

$$\begin{aligned}((-f)+f)(x) &= \frac{1}{f(x)}f(x)\\ &= 1\\ &= 0(x)\end{aligned}$$

(6) Since $f(x)>0$ for all $x$, $\alpha f(x)=[f(x)]^{\alpha}\in V$.

(7)

$$\begin{aligned}\alpha(\beta(f(x))) &= \alpha[f(x)]^{\beta}\\ &= \left[[f(x)]^{\beta}\right]^{\alpha}\\ &= [f(x)]^{\alpha\beta}\\ &= \alpha\beta(f(x))\end{aligned}$$

(8)

$$\begin{aligned}\alpha((f+g)(x)) &= \alpha(f(x)g(x))\\ &= (f(x)g(x))^{\alpha}\\ &= [f(x)]^{\alpha}[g(x)]^{\alpha}\\ &= \alpha f(x)\alpha g(x)\\ &= (\alpha f+\alpha g)(x)\end{aligned}$$

(9)

$$\begin{aligned}(\alpha+\beta)f(x) &= [f(x)]^{\alpha+\beta}\\ &= [f(x)]^{\alpha}[f(x)]^{\beta}\\ &= \alpha f(x)\beta f(x)\\ &= (\alpha f+\beta f)(x)\end{aligned}$$

(10)

$$\begin{aligned} 1f(x) &= [f(x)]^1 \\ &= f(x) \end{aligned}$$

So, this is a vector space.

11. Let $V = M_{nn}$, the vector space of $n \times n$ matrices. Let $W = \{A \in M_{nn} \mid A^T = -A\}$. Determine if $W$ is a subspace of $V$.

Let $A, B \in W$ and let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} (A+B)^T &= A^T + B^T \\ &= -A + (-B) \\ &= -(A+B) \end{aligned}$$

So, $A + B \in W$.

$$\begin{aligned} (\alpha A)^T &= \alpha A^T \\ &= \alpha(-A) \\ &= -(\alpha A) \end{aligned}$$

So, $\alpha A \in W$. Hence, $W$ is a subspace of $V$.

12. Let $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\} \subseteq \mathbb{R}^n$. If $r > n$, show that $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ is linearly dependent.

Since the dimension of $\mathbb{R}^n$ is $n$, the maximum size any linearly independent set can have is $n$. Since $r > n$, it follows that $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ is linearly dependent.

13. Let $V$ be a vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\} \subseteq V$ be a linearly independent set. If $\mathbf{v}_{n+1} \notin$ *span* $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$, show $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n, \mathbf{v}_{n+1}\}$ is also a linearly independent set.

Suppose $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n, \mathbf{v}_{n+1}\}$ is linearly dependent. Then there is some $\mathbf{v}_i$ that can be written as a linear combination of the other vectors, say

$$\mathbf{v}_i = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_{i-1}\mathbf{v}_{i-1} + \alpha_{i+1}\mathbf{v}_{i+1} + \cdots + \alpha_n \mathbf{v}_n + \alpha_{n+1}\mathbf{v}_{n+1}$$

If $\alpha_{n+1} = 0$, then $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a linearly dependent set, which contradicts the assumption that $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is linearly independent. If $\alpha_{n+1} \neq 0$, then we have

$$\mathbf{v}_{n+1} = -\frac{\alpha_1}{\alpha_{n+1}}\mathbf{v}_1 - \frac{\alpha_2}{\alpha_{n+1}}\mathbf{v}_2 - \cdots - \frac{1}{\alpha_{n+1}}\mathbf{v}_i - \cdots - \frac{\alpha_n}{\alpha_{n+1}}\mathbf{v}_n$$

which means $\mathbf{v}_{n+1} \in$ *span* $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$, another contradiction. So, $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n, \mathbf{v}_{n+1}\}$ is a linearly independent set.

14. Let $P_2$ be the vector space of all polynomials of degree less than or equal to 2 (and including the zero polynomial).

 (a) Show $\{1, x, x^2\}$ is a basis for $P_2$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

 Since this matrix is the identity matrix, it follows that $\{1, x, x^2\}$ is a basis for $P_2$.

(b) Is $\{1+x+x^2, x+x^2, x^2\}$ a basis for $P_2$? Why or why not?

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the rref of this matrix is the identity matrix, it follows, that $\{1+x+x^2, x+x^2, x^2\}$ is a basis for $P_2$

(c) Is $\{4+6x+x^2, -1+4x+2x^2, 5+2x-x^2\}$ a basis for $P_2$? Why or why not?

$$\begin{bmatrix} 4 & -1 & 5 \\ 6 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the rref of this matrix is not the identity matrix, it follows, that $\{4+6x+x^2, -1+4x+2x^2, 5+2$
is not a basis for $P_2$

15. Let

$$A = \begin{bmatrix} 1 & -2 & 2 & -1 \\ -3 & 6 & 1 & 10 \\ 1 & -2 & -4 & -7 \end{bmatrix}$$

(a) Determine if $\text{col}(A) = \mathbb{R}^3$.
Row-reducing $A$, we get

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since we do not have a pivot in every row, the vectors cannot span $\mathbb{R}^3$. Hence, $\text{col}(A) \neq \mathbb{R}^3$.

(b) Find a basis for $\text{col}(A)$.
Using the row-reduced matrix, we see that a basis for $\text{col}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ -2 \end{bmatrix} \right\}$$

(c) What is the dimension of $\text{col}(A)$?
Since there are two elements in the basis, we have the dimension of $\text{col}(A)$ is 2.

16. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 5 \\ 6 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 13 \\ 4 \\ 7 \end{bmatrix}$$

(a) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Does it form a basis for $\mathbb{R}^4$? Why or why not?
We construct a matrix by using the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as the columns.

$$\begin{bmatrix} 1 & -1 & 1 \\ 4 & 5 & 13 \\ -1 & 6 & 4 \\ 3 & 2 & 7 \end{bmatrix}$$

Row-reducing, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a pivot in each column, we see the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. However, this set does not form a basis for $\mathbb{R}^4$ since $\mathbb{R}^4$ is 4 dimensional and would need 4 elements in a basis.

(b) Find a basis for $\operatorname{col}(A)$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 5 & 13 \\ -1 & 6 & 4 \\ 3 & 2 & 7 \end{bmatrix}$$

Since the columns of $A$ are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we see that a basis for $\operatorname{col}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 6 \\ 2 \end{bmatrix}; \begin{bmatrix} 1 \\ 13 \\ 4 \\ 7 \end{bmatrix} \right\}$$

(c) Is the vector

$$\begin{bmatrix} -18 \\ -99 \\ 3 \\ 0 \end{bmatrix}$$

in $\operatorname{col}(A)$? Why or why not? If it is, find the coordinates of this vector relative to the basis you found in part b).

To determine if a vector is in $\operatorname{col}(A)$, we need to see if we can solve

$$\begin{bmatrix} 1 & -1 & 1 \\ 4 & 5 & 13 \\ -1 & 6 & 4 \\ 3 & 2 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -18 \\ -99 \\ 3 \\ 0 \end{bmatrix}$$

Augmenting and row-reducing, we get

$$\begin{bmatrix} 1 & -1 & 1 & -18 \\ 4 & 5 & 13 & -99 \\ -1 & 6 & 4 & 3 \\ 3 & 2 & 7 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 117 \\ 0 & 1 & 0 & 66 \\ 0 & 0 & 1 & -69 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that we get the solution

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 117 \\ 66 \\ -69 \end{bmatrix}$$

These are also the coordinates with repsect to the basis.

(d) Is the vector

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

in col $(A)$? Why or why not? If it is, find the coordinates of this vector relative to the basis you found in part b).

Augmenting and row-reducing as we did in part c), we get

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 4 & 5 & 13 & 0 \\ -1 & 6 & 4 & 1 \\ 3 & 2 & 7 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We see this has no solution. Hence, $\mathbf{v}_4 \notin \text{col}(A)$.

(e) Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ a basis for $\mathbb{R}^4$? Why or why not?

Since we have a pivot in every column in the above matrix, and the columns of the matrix are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, the set$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. Since it has 4 elements, it is a basis for $\mathbb{R}^4$.

17. Let

$$A = \begin{bmatrix} 1 & -2 & 2 & -1 \\ -3 & 6 & 1 & 10 \\ 1 & -2 & -4 & -7 \end{bmatrix}$$

(a) Find a basis for row $(A)$ for the matrix $A$ in problem 1.

Row reducing, we have $\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So, a basis for the row space is $\{[1, -2, 0, -3], [0, 0, 1, 1]\}$

(b) Find a basis for row $(A)$ using only original rows of $A$.

Since row $(A) = \text{col}\left(A^T\right)$, we first take $A^T$ and then row reduce

$$\begin{bmatrix} 1 & -3 & 1 \\ -2 & 6 & -2 \\ 2 & 1 & -4 \\ -1 & 10 & -7 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -\frac{11}{7} \\ 0 & 1 & -\frac{6}{7} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, the first two columns correspond to the basis vectors. Hence, a basis for row $(A)$ is $\{[1, -2, 2, -1], [-3, 6, 1, 10]\}$.

(c) What is the dimension of row $(A)$? How does it compare to the dimension of col $(A)$?

Since there are two elements in the basis for row $(A)$, the dimension is 2. After row reducing $A$, we see two columns with pivots. Hence, the dimension of col $(A)$ is 2. So, these two dimensions are equal.

(d) Find a basis for row$(B)$ and col$(B)$ for

$$B = \begin{bmatrix} 1 & 2 & 0 & 3 & 2 & 5 \\ -1 & 1 & -3 & 3 & 4 & 7 \\ 0 & -2 & 2 & -4 & 1 & -3 \\ 2 & 0 & 4 & -2 & 0 & -2 \\ 1 & 0 & 2 & -1 & 1 & 0 \end{bmatrix}$$

Row reducing, we see

$$\begin{bmatrix} 1 & 0 & 2 & -1 & 0 & -1 \\ 0 & 1 & -1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, a basis for the row space is $\{[1,0,2,-1,0,-1], [0,1,-1,2,0,2], [0,0,0,0,1,1]\}$. A basis for the column space is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(e) What is the dimension of row$(B)$ and col$(B)$?

The dimension of both is 3.

18. Let $A$ be an $m \times n$ matrix and $\mathbf{x}$ be an $n \times 1$ column vector.

(a) Show that

$$\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

forms a subspace of $\mathbb{R}^n$ by showing the closure axioms hold.

Let $W = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$, and let $\mathbf{u}, \mathbf{v} \in W$. Then

$$\begin{aligned} A(\mathbf{u} + \mathbf{v}) &= A\mathbf{u} + A\mathbf{v} \\ &= \mathbf{0} + \mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

Hence, $\mathbf{u} + \mathbf{v} \in W$. Also

$$\begin{aligned} A(\alpha\mathbf{u}) &= \alpha(A\mathbf{u}) \\ &= \alpha \cdot \mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

Hence, $\alpha\mathbf{u} \in W$. So, $W$ is a subspace.

(b) For the matrix $B$ of problem 17, solve the equation $B\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$.

Row-reducing, we get

$$\begin{bmatrix} 1 & 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $x_3$, $x_4$, and $x_6$ are the free variables. Let

$$\begin{aligned} x_3 &= r \\ x_4 &= s \\ x_6 &= t \end{aligned}$$

Then

$$\begin{aligned} x_5 &= -t \\ x_2 &= r - 2s - 2t \\ x_1 &= -2r + s + t \end{aligned}$$

Our solutions are

$$\begin{bmatrix} -2r+s+t \\ r-2s-2t \\ r \\ s \\ -t \\ t \end{bmatrix} = r\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(c) Determine a basis that spans null $(A)$.
After writing the solutions in vector form, we have the basis

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(d) What is the dimension of null $(A)$? (This number is called the *nullity* of $A$, denoted nullity$(A)$.)
Since the above basis has three elements, the dimension of null $(B)$ is 3.

(e) What relationship do you observe between rank $(A)$, nullity $(A)$, and the number of columns of $A$?
From problem 3, we found rank $(B) = 3$. We now have nullity $(B) = 3$. The number of columns of $B$ is 6. The relationship is

$$\text{rank}\,(B) + \text{nullity}\,(B) = \text{number of columns of } A$$

19. Show that if $\mathbf{v}$ is orthogonal to both $\mathbf{w}_1$ and $\mathbf{w}_2$, then $\mathbf{v}$ is orthogonal to $k_1\mathbf{w}_1 + k_2\mathbf{w}_2$ for all scalars $k_1$ and $k_2$.

Since $\mathbf{v}$ is orthogonal to both $\mathbf{w}_1$ and $\mathbf{w}_2$, we have $\mathbf{v} \cdot \mathbf{w}_1 = 0 = \mathbf{v} \cdot \mathbf{w}_2$. Now

$$\begin{aligned} \mathbf{v} \cdot (k_1\mathbf{w}_1 + k_2\mathbf{w}_2) &= k_1(\mathbf{v} \cdot \mathbf{w}_1) + k_2(\mathbf{v} \cdot \mathbf{w}_2) \\ &= k_1(0) + k_2(0) \\ &= 0 \end{aligned}$$

20. Find bases for the column space, row space, and nullspace of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

What is the rank and nullity of this matrix?

Row-reducing, we have

$$\begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, a basis for the column space is

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ -5 \\ -9 \end{bmatrix} \right\}$$

a basis for the row space is

$$\{[1, 0, -4, -28, -37, 13], [0, 1, -2, -12, -16, 5]\}$$

and a basis for the null space is

$$\left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 37 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The rank of the matrix is 2, and the nullity is 4.