

MA 303 - Exam 1 Review - Spring 2008

1. The *nullspace* of a matrix A is defined as the set of solutions of the system $A\vec{x} = \vec{0}$.

(a) Find the null space of the matrix $A = \begin{bmatrix} 1 & 2 & -2 & 0 & 1 \\ 2 & 4 & -1 & 0 & -4 \\ -3 & -6 & 12 & 2 & -12 \\ 1 & 2 & -2 & -4 & -5 \end{bmatrix}$.

(b) The vector $\mathbf{v} = \begin{bmatrix} -12 \\ 0 \\ -5 \\ 8 \\ 0 \end{bmatrix}$ is a solution to the system

$$\begin{bmatrix} 1 & 2 & -2 & 0 & 1 \\ 2 & 4 & -1 & 0 & -4 \\ -3 & -6 & 12 & 2 & -12 \\ 1 & 2 & -2 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ -19 \\ -8 \\ -34 \end{bmatrix}$$

(You do NOT need to show this.) Write the set of all solutions of $A\vec{x} = \vec{b}$,

where $\vec{b} = \begin{bmatrix} -2 \\ -19 \\ -8 \\ -34 \end{bmatrix}$

2. Answer the following:

(a) Determine if there exist scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \alpha_4\vec{v}_4 = \vec{b}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -2 \\ 5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -3 \\ -6 \\ 1 \\ -8 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -4 \end{bmatrix}, \vec{b} = \begin{bmatrix} 8 \\ 17 \\ -8 \\ 3 \end{bmatrix}.$$

(b) A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in a vector space V is said to be *linearly independent* if $k_1\vec{v}_1 + k_2\vec{v}_2 + \dots + k_n\vec{v}_n = \vec{0}$ implies $k_1 = k_2 = \dots = k_n = 0$. For example, the vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent in \mathbb{R}^2 because if $k_1\vec{e}_1 + k_2\vec{e}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we get by equating entries that

$$\begin{aligned} 1k_1 + 0k_2 &= 0 \\ 0k_1 + 1k_2 &= 0 \end{aligned}$$

and the only solution to this system is $k_1 = k_2 = 0$.

Determine if the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in part (a) are linearly independent by finding a system of equations, writing the associated augmented matrix, row-reducing, and determining if the zero vector is the ONLY solution to the system.

(c) Determine if the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

are linearly independent by finding a system of equations, writing the associated augmented matrix, row-reducing, and determining if the zero vector is the ONLY solution to the system.

3. Solve the following system, or explain why the following system has no solution.

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 4 \\ -x_1 + x_2 - x_3 + 2x_4 &= 5 \\ 2x_1 - 3x_2 + 2x_3 - 3x_4 &= -1 \\ x_2 - x_4 &= -9 \end{aligned}$$

4. Find conditions on b_1, b_2, b_3 so that

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$$

is consistent.

5. Find (if possible) the inverses of the following matrices. If the matrix has no inverse, explain why it does not.

(a) $A = \begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$

6. Consider the system

$$\begin{aligned} (a - \lambda)x + by &= 0 \\ cx + (d - \lambda)y &= 0 \end{aligned}$$

(a) Can this system ever be inconsistent? Why or why not?

- (b) Show that the values of λ for which the system has nontrivial solutions should satisfy the quadratic equation

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

You may assume that $a - \lambda$ and c are not equal to zero.

7. Let

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Perform the following operations, if possible. If the operation is not possible, state why.

- (a) A^2
 - (b) $(3C^T - A)B$
 - (c) D^2
 - (d) $EA + C^T$
8. Consider the matrix

$$A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$$

- (a) Using elementary row-operations, find a matrix B so that A is row-equivalent to B , and B is in row-echelon form.
 - (b) Is A row-equivalent to the identity matrix? Why or why not?
 - (c) If A is row-equivalent to the identity matrix, find a sequence E_1, E_2, \dots, E_n of elementary matrices so that $E_n E_{n-1} \cdots E_1 = A^{-1}$. (You do NOT need to multiply the matrices together.) If not, explain why A^{-1} does not exist.
9. Consider the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

- (a) Write the system of equations in the form $Ax = b$.
- (b) Write the system of equations as an augmented matrix $[A \mid b]$.
- (c) Using the augmented matrix and row operations, solve the system.

10. Show that the matrix

$$X = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

is singular for any values of the entries a, b, c, d, e, f, g, h . (HINT: Show X is row-equivalent to a matrix with a row of zeros and then explain why X must be singular.)

11. A square matrix A is called *symmetric* if $A^T = A$.

(a) For any matrix A , prove that $A^T A$ is symmetric.

(b) Suppose A is a square matrix such that $A^T A = A$. Use part (a) to prove that A is symmetric.

(c) Use part (b) to show if $A = A^T A$, then A is *idempotent*; that is, prove that if $A = A^T A$, then $A^2 = A$.

12. A matrix $A = [a_{ij}]$ is *lower triangular* if all the entries above the main diagonal are 0. Notationally, we have $a_{ij} = 0$ whenever $i < j$. Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are $n \times n$ upper triangular matrices. By appealing to the elements of these matrices, prove that AB is also a lower triangular matrix; in other words, if $AB = [c_{ij}]$, show $c_{ij} = 0$ if $i < j$.

13. For a square matrix A , prove that $(A^k)^T = (A^T)^k$ for all integers $k \geq 1$.