Vector Space Practice

Directions: Determine whether or not the following are vector spaces.

1. $\mathbb{R}^3$ under ordinary addition, but scalar mulltiplication is defined as $k(x,y,z) = (kx,y,z)$.

   (A) Let $(x,y,z), (x',y',z') \in \mathbb{R}^3$. Then,

   $$(x,y,z) + (x',y',z') = (x+x', y+y', z+z') \in \mathbb{R}^3$$

   So, axiom A holds.

   (S)

   $$k(x,y,z) = (kx,y,z) \in \mathbb{R}^3$$

   So, axiom S holds.

   (A1)

   $$\begin{aligned} (x,y,z) + ((x',y',z') + (x'',y'',z'')) &= (x,y,z) + (x'+x'', y'+y'', z'+z'') \\ &= (x + (x'+x''), y + (y'+y''), z + (z'+z'')) \\ &= ((x+x') + x'', (y+y') + y'', (z+z') + z'') \\ &= (x+x', y+y', z+z') + (x'',y'',z'') \\ &= ((x,y,z) + (x',y',z')) + (x'',y'',z'') \end{aligned}$$

   So, axiom A1 holds.

   (A2) Let $\mathbf{0} = (0,0,0)$. Then

   $$\begin{aligned} (x,y,z) + \mathbf{0} &= (x,y,z) + (0,0,0) \\ &= (x,y,z) \\ &= (0,0,0) + (x,y,z) \\ &= \mathbf{0} + (x,y,z) \end{aligned}$$

   So, axiom A2 holds.

   (A3) If we have (x,y,z), consider $-(x,y,z) = (-x,-y,-z)$. Then,

   $$\begin{aligned} (x,y,z) + (-(x,y,z)) &= (x,y,z) + (-x,-y,-z) \\ &= (x + (-x), y + (-y), z + (-z)) \\ &= (0,0,0) \\ &= \mathbf{0} \end{aligned}$$

   So, axiom A3 holds.

   (A4)

   $$\begin{aligned} (x,y,z) + (x',y',z') &= (x+x', y+y', z+z') \\ &= (x'+x, y'+y, z'+z) \\ &= (x',y',z') + (x,y,z) \end{aligned}$$

   So, axiom A4 holds.

(S1)

$$\begin{aligned}
k\left((x,y,z)+\left(x',y',z'\right)\right) &= k\left(x+x',y+y',z+z'\right) \\
&= \left(k\left(x+x'\right),y+y',z+z'\right) \\
&= \left(kx+kx',y+y',z+z'\right) \\
&= (kx,y,z)+\left(kx',y',z'\right) \\
&= k(x,y,z)+k\left(x',y',z'\right)
\end{aligned}$$

So, axiom S1 holds.

(S2) Let $\alpha=\beta=1,(x,y,z)=(1,1,1)$. Then

$$\begin{aligned}
(\alpha+\beta)(x,y,z) &= 2(1,1,1) \\
&= (2,1,1)
\end{aligned}$$

But

$$\begin{aligned}
\alpha(x,y,z)+\beta(x,y,z) &= 1(1,1,1)+1(1,1,1) \\
&= (1,1,1)+(1,1,1) \\
&= (2,2,2)
\end{aligned}$$

So, axiom S2 fails.

(S3)

$$\begin{aligned}
\alpha(\beta(x,y,z)) &= \alpha(\beta x,y,z) \\
&= (\alpha(\beta x),y,z) \\
&= ((\alpha\beta)x,y,z) \\
&= (\alpha\beta)(x,y,z)
\end{aligned}$$

So, axiom S3 holds

(S4) Since $1(x,y,z)=(x,y,z)$, axiom S4 holds.
This is not a vector space.

2. The set M_{22} of all 2×2 matrices under ordinary scalar multiplication, but addition defined as $A+B=\mathbf{0}$, the zero matrix, for all $A,B\in M_{22}$.

(A)

$$A+B=\mathbf{0}\in M_{22}$$

So, axiom A holds.

(S)

$$\alpha A\in M_{22}$$

So, axiom S holds.

(A1)

$$\begin{aligned} A + (B + C) &= A + \mathbf{0} \\ &= \mathbf{0} \\ &= \mathbf{0} + C \\ &= (A + B) + C \end{aligned}$$

So, axiom A1 holds.

(A2) If $\mathbf{0} = B$ and A is not the zero matrix, then $A + B = \mathbf{0} \neq A$. So, axiom A2 fails.

(A3) Since axiom A2 fails, so must axiom A3.

(A4)

$$A + B = \mathbf{0} = B + A$$

So, axiom A4 holds.

(S1)

$$\begin{aligned} \alpha (A + B) &= \alpha \mathbf{0} \\ &= \mathbf{0} \\ &= \alpha A + \alpha B \end{aligned}$$

So, axiom S1 holds.

(S2) Let $\alpha = \beta = 1$, and let A be any non-zero matrix. Then, $(\alpha + \beta) A = 2A$, but $\alpha A + \beta A = \mathbf{0}$. Hence, axiom S2 fails.

(S3)

$$\alpha (\beta A) = (\alpha \beta) A$$

So, axiom S3 holds. (We know this is true for matrix multiplication.)

(S4) Since $1A = A$, axiom S4 holds.
This is not a vector space.

3. The set $F(-\infty, \infty)$ of all functions on the real line with ordinary scalar multiplication and addition defined as $(f + g)(x) = \max\{f(x), g(x)\}$.

(A) Since $(f + g)(x) = \max\{f(x), g(x)\}$ is a function, $(f + g)(x) \in F(-\infty, \infty)$, and axiom A holds.

(S) $(\alpha f)(x) = \alpha f(x) \in F(-\infty, \infty)$. So, axiom S holds.

(A1)

$$\begin{aligned} ((f + g) + h)(x) &= \max\{(f + g)(x), h(x)\} \\ &= \max\{\max\{f(x), g(x)\}, h(x)\} \\ &= \max\{f(x), g(x), h(x)\} \\ &= \max\{f(x), \max\{g(x), h(x)\}\} \\ &= \max\{f(x), (g + h)(x)\} \\ &= (f + (g + h))(x) \end{aligned}$$

So, axiom A1 holds.

(A2) If $\mathbf{0} = g(x)$, then $f(x) + g(x) = \mathbf{0}$ for all x. If we define

$$f(x) = \begin{cases} g(x) - 1, & x < 0 \\ g(x) + 1, & x \geq 0 \end{cases}$$

Then

$$\begin{aligned} (f+g)(x) &= \begin{cases} g(x), & x < 0 \\ f(x), & x \geq 0 \end{cases} \\ &\neq f(x) \end{aligned}$$

So, no matter what we try to define $\mathbf{0}$ as, we can always find a function $f(x)$ so that $f(x) + \mathbf{0} \neq f(x)$ for all x. Hence, axiom A2 fails.

(A3) Axiom A3 fails - there is no zero vector, so we cannot have an inverse.

(A4)

$$\begin{aligned} (f+g)(x) &= \max\{f(x), g(x)\} \\ &= \max\{g(x), f(x)\} \\ &= (g+f)(x) \end{aligned}$$

So, axiom A4 holds

(S1) Let $f(x) = 1$, $g(x) = 3$, $\alpha = -1$. Then,

$$\begin{aligned} \alpha(f+g)(x) &= \alpha \max\{f(x), g(x)\} \\ &= (-1) \cdot 3 \\ &= -3 \end{aligned}$$

But,

$$\begin{aligned} (\alpha f + \alpha g)(x) &= \max\{\alpha f(x), \alpha g(x)\} \\ &= \max\{-1, -3\} \\ &= -1 \end{aligned}$$

So, axiom S1 fails.

(S2) Let $f(x) = 1$, $\alpha = 1$, $\beta = -1$. Then

$$\begin{aligned} (\alpha + \beta) f(x) &= 0 f(x) \\ &= 0 \end{aligned}$$

But,

$$\begin{aligned} (\alpha f + \beta f)(x) &= \max\{\alpha f(x), \beta f(x)\} \\ &= \max\{1, -1\} \\ &= 1 \end{aligned}$$

So, axiom S2 fails.

(S3)

$$\begin{aligned}((\alpha\beta) f)(x) &= (\alpha\beta) f(x) \\ &= \alpha (\beta f)(x) \\ &= (\alpha (\beta f))(x)\end{aligned}$$

So, axiom S3 holds

(S4)

$$\begin{aligned}(1f)(x) &= 1 \cdot f(x) \\ &= f(x)\end{aligned}$$

So, axiom S4 holds

This is not a vector space.

4. The set of all 3×3 matrices of the form

$$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}$$

where a, b, c, d, e are real numbers, with ordinary addition and scalar multiplication.

(A)

$$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} + \begin{bmatrix} f & 0 & g \\ 0 & h & 0 \\ i & 0 & j \end{bmatrix} = \begin{bmatrix} a+f & 0 & b+g \\ 0 & c+h & 0 \\ d+i & 0 & e+j \end{bmatrix}$$

which has the proper form. Hence, axiom A holds.

(S)

$$\begin{aligned}\alpha A &= \alpha \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} \\ &= \begin{bmatrix} \alpha a & 0 & \alpha b \\ 0 & \alpha c & 0 \\ \alpha d & 0 & \alpha e \end{bmatrix}\end{aligned}$$

which is of the proper form. So, axiom S holds.

(A1)

$$
\begin{aligned}
\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} + \left(\begin{bmatrix} f & 0 & g \\ 0 & h & 0 \\ i & 0 & j \end{bmatrix} + \begin{bmatrix} k & 0 & l \\ 0 & m & 0 \\ n & 0 & o \end{bmatrix} \right) &= \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} + \begin{bmatrix} f+k & 0 & g+l \\ 0 & h+m & 0 \\ i+n & 0 & j+o \end{bmatrix} \\
&= \begin{bmatrix} a+(f+k) & 0 & b+(g+l) \\ 0 & c+(h+m) & 0 \\ d+(i+n) & 0 & e+(j+o) \end{bmatrix} \\
&= \begin{bmatrix} (a+f)+k & 0 & (b+g)+l \\ 0 & (c+h)+m & 0 \\ (d+i)+n & 0 & (e+j)+o \end{bmatrix} \\
&= \begin{bmatrix} a+f & 0 & b+g \\ 0 & c+h & 0 \\ d+i & 0 & e+j \end{bmatrix} + \begin{bmatrix} k & 0 & l \\ 0 & m & 0 \\ n & 0 & o \end{bmatrix} \\
&= \left(\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} + \begin{bmatrix} f & 0 & g \\ 0 & h & 0 \\ i & 0 & j \end{bmatrix} \right) + \begin{bmatrix} k & \\ 0 & \\ n & \end{bmatrix}
\end{aligned}
$$

So, axiom A1 holds.

(A2) Let

$$
\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Then

$$
\begin{aligned}
\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} + \mathbf{0} &= \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}
\end{aligned}
$$

So, axiom A2 holds

(A3) If

$$
A = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}
$$

let

$$
(-A) = \begin{bmatrix} -a & 0 & -b \\ 0 & -c & 0 \\ -d & 0 & -e \end{bmatrix}
$$

Then

$$\begin{aligned} A + (-A) &= \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} + \begin{bmatrix} -a & 0 & -b \\ 0 & -c & 0 \\ -d & 0 & -e \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -a & 0 & -b \\ 0 & -c & 0 \\ -d & 0 & -e \end{bmatrix} + \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} \\ &= (-A) + A \end{aligned}$$

So, axiom A3 holds.

(A4)

$$\begin{aligned} \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} + \begin{bmatrix} f & 0 & g \\ 0 & h & 0 \\ i & 0 & j \end{bmatrix} &= \begin{bmatrix} a+f & 0 & b+g \\ 0 & c+h & 0 \\ d+i & 0 & e+j \end{bmatrix} \\ &= \begin{bmatrix} f+a & 0 & g+b \\ 0 & h+c & 0 \\ i+d & 0 & j+e \end{bmatrix} \\ &= \begin{bmatrix} f & 0 & g \\ 0 & h & 0 \\ i & 0 & j \end{bmatrix} + \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} \end{aligned}$$

So, axiom A4 holds.

(S1)

$$\begin{aligned}
\alpha\left(A+B\right) &= \alpha\left(\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} + \begin{bmatrix} f & 0 & g \\ 0 & h & 0 \\ i & 0 & j \end{bmatrix}\right) \\
&= \alpha\begin{bmatrix} a+f & 0 & b+g \\ 0 & c+h & 0 \\ d+i & 0 & e+j \end{bmatrix} \\
&= \begin{bmatrix} \alpha\left(a+f\right) & 0 & \alpha\left(b+g\right) \\ 0 & \alpha\left(c+h\right) & 0 \\ \alpha\left(d+i\right) & 0 & \alpha\left(e+j\right) \end{bmatrix} \\
&= \begin{bmatrix} \alpha a+\alpha f & 0 & \alpha b+\alpha g \\ 0 & \alpha c+\alpha h & 0 \\ \alpha d+\alpha i & 0 & \alpha e+\alpha j \end{bmatrix} \\
&= \begin{bmatrix} \alpha a & 0 & \alpha b \\ 0 & \alpha c & 0 \\ \alpha d & 0 & \alpha e \end{bmatrix} + \begin{bmatrix} \alpha f & 0 & \alpha g \\ 0 & \alpha h & 0 \\ i\alpha & 0 & \alpha j \end{bmatrix} \\
&= \alpha\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} + \alpha\begin{bmatrix} f & 0 & g \\ 0 & h & 0 \\ i & 0 & j \end{bmatrix} \\
&= \alpha A+\alpha B
\end{aligned}$$

So, axiom S1 holds.

(S2)

$$\begin{aligned}
\left(\alpha+\beta\right)A &= \left(\alpha+\beta\right)\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} \\
&= \begin{bmatrix} \left(\alpha+\beta\right)a & 0 & \left(\alpha+\beta\right)b \\ 0 & \left(\alpha+\beta\right)c & 0 \\ \left(\alpha+\beta\right)d & 0 & \left(\alpha+\beta\right)e \end{bmatrix} \\
&= \begin{bmatrix} \alpha a+\beta a & 0 & \alpha b+\beta b \\ 0 & \alpha c+\beta c & 0 \\ \alpha d+\beta d & 0 & \alpha e+\beta e \end{bmatrix} \\
&= \begin{bmatrix} \alpha a & 0 & \alpha b \\ 0 & \alpha c & 0 \\ \alpha d & 0 & \alpha e \end{bmatrix} + \begin{bmatrix} \beta a & 0 & \beta b \\ 0 & \beta c & 0 \\ \beta d & 0 & \beta e \end{bmatrix} \\
&= \alpha\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} + \beta\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} \\
&= \alpha A+\beta A
\end{aligned}$$

So, axiom S2 holds.

(S3)

$$\begin{aligned} \alpha(\beta A) &= \alpha\left(\beta\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}\right) \\ &= \alpha\begin{bmatrix} \beta a & 0 & \beta b \\ 0 & \beta c & 0 \\ \beta d & 0 & \beta e \end{bmatrix} \\ &= \begin{bmatrix} \alpha\beta a & 0 & \alpha\beta b \\ 0 & \alpha\beta c & 0 \\ \alpha\beta d & 0 & \alpha\beta e \end{bmatrix} \\ &= (\alpha\beta)\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} \\ &= (\alpha\beta) A \end{aligned}$$

So, axiom S3 holds

(S4) Since $1A = A$ for any matrix A, axiom S4 holds.

This is a vector space.

5. The set M_{22} of all invertible 2×2 matrices under ordinary scalar multiplication, but addition is defined as $A + B = AB$.

(A)

$$A + B = AB \in M_{22}$$

So, axiom A holds.

(S)

$$\alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} \in M_{22}$$

So, axiom S holds.

(A1)

$$\begin{aligned} A + (B + C) &= A + BC \\ &= A(BC) \\ &= (AB)C \\ &= (A + B)C \\ &= (A + B) + C \end{aligned}$$

So, axiom A1 holds

(A2) Let

$$\mathbf{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$\begin{aligned} A+\mathbf{0} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \mathbf{0} + A \end{aligned}$$

So, axiom A2 holds.

(A3) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

We need to find a matrix B such that $A+B=\mathbf{0}$, or $AB=I$. This says $B=A^{-1}$, but A^{-1} does not exist. So, axiom A3 fails.

(A4) Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 \\ 2 & 5 \end{bmatrix}$$

Then

$$\begin{aligned} A+B &= AB \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 14 \\ 8 & 19 \end{bmatrix} \end{aligned}$$

But,

$$\begin{aligned} B+A &= BA \\ &= \begin{bmatrix} 2 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 16 \\ 7 & 19 \end{bmatrix} \end{aligned}$$

So, axiom A4 fails.

(S1) Let $\alpha = 2$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B$$

Then,

$$\begin{aligned}\alpha(A+B) &= \alpha(AB)\\ &= 2\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\\ &= \begin{bmatrix}2&0\\0&2\end{bmatrix}\end{aligned}$$

But,

$$\begin{aligned}\alpha A+\alpha B &= 2\begin{bmatrix}1&0\\0&1\end{bmatrix}+2\begin{bmatrix}1&0\\0&1\end{bmatrix}\\ &= \begin{bmatrix}2&0\\0&2\end{bmatrix}+\begin{bmatrix}2&0\\0&2\end{bmatrix}\\ &= \begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}2&0\\0&2\end{bmatrix}\\ &= \begin{bmatrix}4&0\\0&4\end{bmatrix}\end{aligned}$$

So, axiom S1 fails.

(S2) Let $\alpha = 1, \beta = -1$,

$$A = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Then,

$$\begin{aligned}(\alpha+\beta)A &= 0\begin{bmatrix}1&0\\0&1\end{bmatrix}\\ &= \begin{bmatrix}0&0\\0&0\end{bmatrix}\end{aligned}$$

But,

$$\begin{aligned}\alpha A+\beta A &= 1\begin{bmatrix}1&0\\0&1\end{bmatrix}+(-1)\begin{bmatrix}1&0\\0&1\end{bmatrix}\\ &= \begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\\ &= \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\\ &= \begin{bmatrix}-1&0\\0&-1\end{bmatrix}\end{aligned}$$

So, axiom S2 fails.

(S3)

$$\alpha(\beta A) = (\alpha\beta)A$$

So, axiom S3 holds.

(S4) Since $1A = A$, axiom S4 holds.

This is not a vector space.