1. The *nullspace* of a matrix \( A \) is defined as the set of solutions of the system \( A \vec{x} = \vec{0} \).

(a) Find the null space of the matrix \( A = \begin{bmatrix} 1 & 2 & -2 & 0 & 1 \\ 2 & 4 & -1 & 0 & -4 \\ -3 & -6 & 12 & 2 & -12 \\ 1 & 2 & -2 & -4 & -5 \end{bmatrix} \).

(b) The vector \( \vec{v} = \begin{bmatrix} -12 \\ 0 \\ -5 \\ 8 \\ 0 \end{bmatrix} \) is a solution to the system
\[
\begin{bmatrix} 1 & 2 & -2 & 0 & 1 \\ 2 & 4 & -1 & 0 & -4 \\ -3 & -6 & 12 & 2 & -12 \\ 1 & 2 & -2 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ -19 \\ -8 \\ -34 \end{bmatrix}
\]

(You do NOT need to show this.) Write the set of all solutions of \( A \vec{x} = \vec{b} \), where \( \vec{b} = \begin{bmatrix} -2 \\ -19 \\ -8 \\ -34 \end{bmatrix} \).

2. Answer the following:

(a) Determine if there exist scalars \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) such that \( \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 + \alpha_4 \vec{v}_4 = \vec{b} \), where
\[
\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -2 \\ 5 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -3 \\ -6 \\ 1 \\ -8 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 8 \\ 17 \\ -8 \\ 3 \end{bmatrix}.
\]

(b) A set of vectors \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \) in a vector space \( V \) is said to be *linearly independent* if \( k_1 \vec{v}_1 + k_2 \vec{v}_2 + \cdots + k_n \vec{v}_n = \vec{0} \) implies \( k_1 = k_2 = \cdots = k_n = 0 \). For example, the vectors \( \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) are linearly independent in \( \mathbb{R}^2 \) because if \( k_1 \vec{e}_1 + k_2 \vec{e}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), we get by equating entries that
\[
1k_1 + 0k_2 = 0 \\
0k_1 + 1k_2 = 0
\]
and the only solution to this system is \( k_1 = k_2 = 0 \).

Determine if the vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \) in part (a) are linearly independent by finding a system of equations, writing the associated augmented matrix, row-reducing, and determining if the zero vector is the ONLY solution to the system.

(c) Determine if the vectors

\[
\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}
\]

are linearly independent by finding a system of equations, writing the associated augmented matrix, row-reducing, and determining if the zero vector is the ONLY solution to the system.

3. Solve the following system, or explain why the following system has no solution.

\[
\begin{align*}
x_1 - 2x_2 + x_3 - x_4 &= 4 \\
x_1 - x_2 - x_3 + 2x_4 &= 5 \\
2x_1 - 3x_2 + 2x_3 - 3x_4 &= -1 \\
x_2 - x_4 &= -9
\end{align*}
\]

4. Find conditions on \( b_1, b_2, b_3 \) so that

\[
\begin{align*}
x_1 - 2x_2 + 5x_3 &= b_1 \\
x_2 - 5x_2 + 8x_3 &= b_2 \\
-3x_1 + 3x_2 - 3x_3 &= b_3
\end{align*}
\]

is consistent.

5. Find (if possible) the inverses of the following matrices. If the matrix has no inverse, explain why it does not.

(a) \( A = \begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix} \)

(b) \( B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix} \)

6. Consider the system

\[
(a - \lambda)x + by = 0 \\
ax + (d - \lambda)y = 0
\]

(a) Can this system ever be inconsistent? Why or why not?
(b) Show that the values of $\lambda$ for which the system has nontrivial solutions should satisfy the quadratic equation

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

You may assume that $a - \lambda$ and $c$ are not equal to zero.

7. Let

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Perform the following operations, if possible. If the operation is not possible, state why.

(a) $A^2$
(b) $(3C^T - A)B$
(c) $D^2$
(d) $EA + C^T$

8. Consider the matrix

$$A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$$

(a) Using elementary row-operations, find a matrix $B$ so that $A$ is row-equivalent to $B$, and $B$ is in row-echelon form.

(b) Is $A$ row-equivalent to the identity matrix? Why or why not?

(c) If $A$ is row-equivalent to the identity matrix, find a sequence $E_1, E_2, ..., E_n$ of elementary matrices so that $E_nE_{n-1} \cdots E_1 = A^{-1}$. (You do NOT need to multiply the matrices together.) If not, explain why $A^{-1}$ does not exist.

9. Consider the system of equations

$$x_1 + x_2 + 2x_3 = 8$$
$$-x_1 - 2x_2 + 3x_3 = 1$$
$$3x_1 - 7x_2 + 4x_3 = 10$$

(a) Write the system of equations in the form $Ax = b$.

(b) Write the system of equations as an augmented matrix $[A \mid b]$.

(c) Using the augmented matrix and row operations, solve the system.
10. Show that the matrix
\[
X = \begin{bmatrix}
0 & a & 0 & 0 \\
b & 0 & c & 0 \\
d & e & 0 & g \\
f & 0 & g & 0 \\
h & 0 & 0 & 0
\end{bmatrix}
\]
is singular for any values of the entries \(a, b, c, d, e, f, g, h\). (HINT: Show \(X\) is row-equivalent to a matrix with a row of zeros and then explain why \(X\) must be singular.)

11. A square matrix \(A\) is called symmetric if \(A^T = A\).

(a) For any matrix \(A\), prove that \(A^T A\) is symmetric.
(b) Suppose \(A\) is a square matrix such that \(A^T A = A\). Use part (a) to prove that \(A\) is symmetric.
(c) Use part (b) to show if \(A = A^T A\), then \(A\) is idempotent; that is, prove that if \(A = A^T A\), then \(A^2 = A\).

12. A matrix \(A = [a_{ij}]\) is lower triangular if all the entries above the main diagonal are 0. Notationally, we have \(a_{ij} = 0\) whenever \(i < j\). Suppose \(A = [a_{ij}]\) and \(B = [b_{ij}]\) are \(n \times n\) upper triangular matrices. By appealing to the elements of these matrices, prove that \(AB\) is also a lower triangular matrix; in other words, if \(AB = [c_{ij}]\), show \(c_{ij} = 0\) if \(i < j\).

13. For a square matrix \(A\), prove that \((A^k)^T = (A^T)^k\) for all integers \(k \geq 1\).